AN ARC-LIKE CONTINUUM THAT ADMITS A HOMEOMORPHISM WITH ENTROPY FOR ANY GIVEN VALUE

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Abstract. An arc-like continuum \( X \) is constructed with the following properties: (1) for every \( \epsilon \in [0, \infty] \) there exists a homeomorphism \( g_\epsilon : X \rightarrow X \) such that the entropy of \( g_\epsilon \) is \( \epsilon \) and (2) \( X \) does not contain a pseudo-arc. This answers a question by W. Lewis.

1. Introduction

Let \( f : X \rightarrow X \) be a continuous function. The topological entropy of \( f \) is a number in \([0, \infty]\) that measures the rate of expansion and mixing of \( f \). For more on entropy see [1], [8] and [9]. The purpose of this work is to construct an arc-like continuum \( X \) such that for every \( \epsilon \in [0, \infty] \) there exists a homeomorphism \( g_\epsilon : X \rightarrow X \) such that the entropy of \( g_\epsilon \) is \( \epsilon \) and such that \( X \) contains no pseudo-arc. This answers a question by Wayne Lewis in the negative (see [4]).

A continuum is a compact, connected metric space. A continuum is arc-like if it is the inverse limit of arcs. Arc-like continua are also called chainable. A continuum \( X \) is decomposable if there exist proper subcontinua \( A \) and \( B \) such that \( X = A \cup B \). A continuum is indecomposable if it is not decomposable. A continuum is hereditarily indecomposable if every subcontinuum is indecomposable. The pseudo-arc is an arc-like hereditarily indecomposable continuum. For more on the pseudo-arc see [4]. It has been shown in [6] that every arc-like continuum that admits an homeomorphism with positive entropy must contain an indecomposable subcontinuum. Also, the pseudo-arc admits an continuum-wise expansive homeomorphism which implies that it admits an infinite entropy homeomorphism (see [3] and [7]). However, it is unknown to whether the pseudo-arc admits a positive but finite entropy homeomorphism.

2. Preliminaries on Entropy and Inverse Limits

\( U \) is a cover of \( X \) if \( X \subset \bigcup_{U \in U} U \). \( U \) is an open cover if every element of \( U \) is open. Let \( X \) be a compact metric space, \( f : X \rightarrow X \) be a map, and \( U \) be a finite cover of \( X \). Define \( N(U, X) \) be the number of sets in a subcover of \( U \) on \( X \) with smallest cardinality. If \( U \) and \( V \) are two open covers of \( X \), let \( U \lor V = \{ U \cap V | U \in U, V \in V \} \) and \( f^{-1}(U) = \{ f^{-1}(U) | U \in U \} \). Also, define

\[
\bigvee_{i=0}^{n-1} f^{-i}(U) = U \lor f^{-1}(U) \lor ... \lor f^{-n+1}(U), \text{ where } f^0 = \text{id}
\]

and

\[
\text{Ent}(f, U) = \lim_{n \rightarrow \infty} (1/n) \log N(\bigvee_{i=0}^{n-1} f^{-i}(U), X).
\]

Then the topological entropy of \( f \) is defined as

\[
\text{Ent}(f) = \sup\{ \text{Ent}(f, U, X) : U \text{ is an open cover of } X \}.
\]

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If \( \mathcal{U} \) and \( \mathcal{V} \) are finite covers of \( X \) such that for every \( V \in \mathcal{V} \) there exists a \( U \in \mathcal{U} \) such that \( V \subset U \), then \( \mathcal{V} \) refines \( \mathcal{U} \). If for every \( V \in \mathcal{V} \) there exists a \( U \in \mathcal{U} \) such that \( \overline{V} \subset U \), then \( \mathcal{V} \) closure refines \( \mathcal{U} \).

The following propositions are well known and can be found in several texts such as [8].

**Proposition 1.** If \( \mathcal{V}, \mathcal{U} \) are covers of \( X \) such that \( \mathcal{V} \) refines \( \mathcal{U} \), then \( \text{Ent}(f, \mathcal{V}) \geq \text{Ent}(f, \mathcal{U}) \).

**Proposition 2.** For each positive integer \( k \), \( \text{Ent}(f^k) = k \text{Ent}(f) \).

Let \( T : [0, 1] \to [0, 1] \) be defined by

\[
T(x) = \begin{cases} 
2x & \text{if } x \in [0, 1/2] \\
2 - 2x & \text{if } x \in (1/2, 1],
\end{cases}
\]

\( T(x) \) is often called the *tent map*.

The following theorem is well-known. For further details see [1]:

**Theorem 3.** \( \text{Ent}(T) = \log(2) \).

**Theorem 4.** Suppose that \( f, g, \) and \( h \) are continuous functions such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{h} & & \downarrow{h} \\
Y & \xrightarrow{g} & Y
\end{array}
\]

then \( \text{Ent}(g) \leq \text{Ent}(f) \).

*Proof.* Follows from the fact that if \( \mathcal{U} \) is a finite open cover of \( Y \) then \( \text{Ent}(f, h^{-1}(\mathcal{U})) = \text{Ent}(g, \mathcal{U}) \).

**Corollary 5.** Suppose that \( f \) and \( g \) are continuous functions and \( h \) is a homeomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{h} & & \downarrow{h} \\
Y & \xrightarrow{g} & Y
\end{array}
\]

then \( \text{Ent}(g) = \text{Ent}(f) \).

Let \( I \) be the unit interval and \( f_i : I \to I \) be a continuous function called a *bonding map*. The collection \( \{I, f_i\}_{i=1}^{\infty} \) is called an inverse system. Each interval \( I \) is called a factor space of the inverse system. If each bonding map is the same map \( f \), then the inverse system can be written as \( \{I, f\}_{i=1}^{\infty} \). Every inverse system \( \{I, f_i\}_{i=1}^{\infty} \) determines a topological space \( X \) called the inverse limit of the system and is written \( X = \lim_{\leftarrow \substack{i=1}} \{I, f_i\}_{i=1}^{\infty} \). The space \( X \) is the subspace of the Cartesian product \( \prod_{i=1}^{\infty} I \) given by

\[
X = \lim_{\leftarrow \substack{i=1}} \{I, f_i\}_{i=1}^{\infty} = \{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} I | f_i(x_{i+1}) = x_i \}.
\]

\( X \) has the subspace topology induced on it by \( \prod_{i=1}^{\infty} I \). If \( x = (x_i)_{i=1}^{\infty} \) and \( y = (y_i)_{i=1}^{\infty} \) are two points of the inverse limit, we define distance to be

\[
d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.
\]
For more on the inverse limit arcs see [2].

If each bonding map \( f_i \) is the same map \( f \) then there is a shift homeomorphism \( \hat{f} : X \rightarrow X \) induced by \( f \):

\[
\hat{f}(x) = \hat{f}(x_1, x_2, x_3\ldots) = (f(x_1), f(x_2), f(x_3), \ldots) = (f(x_1), x_1, x_2, \ldots).
\]

Also, notice that

\[
\hat{f}^{-1}(x_1, x_2, x_3\ldots) = (x_2, x_3, x_4\ldots).
\]

The following theorem aids in the computation of entropy of functions on inverse limits and is found in [9]:

**Theorem 6.** Let \( Y \) be a compact Hausdorff space, \( f : Y \rightarrow Y \) be a continuous onto function and \( X = \lim_{i=1}^\infty \{Y, f\} \). Suppose that \( g : Y \rightarrow Y \) is continuous and satisfies \( g \circ f = f \circ g \). Additionally, suppose that \( \hat{g} : X \rightarrow X \) is defined by \( \hat{g}(i) = (g(i)) \). Then Ent(\( \hat{g} \)) = \( \text{Ent}(g) \).

The following theorem by Mahavier [5] gives a sufficient condition for when each subcontinuum in the inverse limit space must contain an arc.

**Theorem 7.** If \( f \) is a piecewise monotone function from \([0, 1]\) onto \([0, 1]\), then each subcontinuum of \( X = \lim_{i=1}^\infty \{0, 1, f\} \) must contain an arc.

We will use the following corollary to show that are continuum in the main result contains no pseudo-arc.

**Corollary 8.** If \( f \) is a piecewise monotone function from \([0, 1]\) onto \([0, 1]\), then \( X = \lim_{i=1}^\infty \{0, 1, f\} \) contains no pseudo-arc.

**Proof.** Suppose that \( X \) contains a pseudo-arc \( P \). Then by Theorem 7, \( P \) must contain an arc. However, arcs are decomposable, so \( P \) is not hereditarily indecomposable and hence not a pseudo-arc. \( \square \)

### 3. Main Result

In this section the main result is constructed. If \( \mathcal{U} \) is a cover of \( X \) and \( Y \subset X \) then define \( \mathcal{U}(Y) = \{U \cap Y | U \in \mathcal{U} \text{ and } U \cap Y \neq \emptyset\} \).

**Lemma 9.** Suppose that \( h : X \rightarrow X \) is a continuous function and \( Y \subset X \) such that \( h(Y) = Y \). Then \( \text{Ent}(h|_Y) \leq \text{Ent}(h) \).

**Proof.** Let \( \mathcal{U} \) be a finite open cover of \( Y \) (relative to the subspace topology induced by \( X \)). Then there exists a finite open cover \( \mathcal{V} \) of \( X \) such that \( \mathcal{V}(Y) = \mathcal{U} \). Hence for each \( n \),

\[
N(\bigvee_{i=0}^{n-1} (h|_Y)^{-i}(\mathcal{U}), Y) = N(\bigvee_{i=0}^{n-1} (h|_Y)^{-i}(\mathcal{V}(Y)), Y) \leq N(\bigvee_{i=0}^{n-1} h^{-i}(\mathcal{V}), X).
\]

Thus, \( \text{Ent}(h|_Y, \mathcal{U}) \leq \text{Ent}(h, \mathcal{V}) \) and the result follows. \( \square \)

**Lemma 10.** For \( a \in (0, 1) \) let \( f : [0, a] \rightarrow [0, a] \) and \( g : [a, 1] \rightarrow [a, 1] \) be continuous functions such that \( f(a) = g(a) \). Define

\[
h(x) = \begin{cases} 
  f(x) & \text{if } x \in [0, a] \\
  g(x) & \text{if } x \in [a, 1],
\end{cases}
\]

then \( \text{Ent}(h) = \max\{\text{Ent}(f), \text{Ent}(g)\} \).
Proof. Let $\mathcal{U}$ be any finite open cover of $[0,1]$. Then there exists finite open covers $\mathcal{V}_0$ and $\mathcal{V}_1$ of $[0,a]$ and $[a,1]$ respectively such that $\mathcal{V}_0 \cup \mathcal{V}_1$ refines $\mathcal{U}$. Then
\[
N(\bigvee_{i=0}^{n-1} (h)^{-i}(\mathcal{V}_0 \cup \mathcal{V}_1), [0,1]) = N(\bigvee_{i=0}^{n-1} (f)^{-i}(\mathcal{V}_0), [0,a]) + N(\bigvee_{i=0}^{n-1} (g)^{-i}(\mathcal{V}_1)[a,1])
\]
\[
\leq 2 \max\{N(\bigvee_{i=0}^{n-1} (f)^{-i}(\mathcal{V}_0), [0,a]), N(\bigvee_{i=0}^{n-1} (g)^{-i}(\mathcal{V}_1), [a,1])\}.
\]
So
\[
\text{Ent}(h, \mathcal{U}) \leq \text{Ent}(h, \mathcal{V}_0 \cup \mathcal{V}_1)
\]
\[
= \limsup_{n \to \infty} \frac{\log N(\bigvee_{i=0}^{n-1} (h)^{-i}(\mathcal{V}_0 \cup \mathcal{V}_1), [0,1])}{n}
\]
\[
\leq \limsup_{n \to \infty} \frac{\log(2 \max\{N(\bigvee_{i=0}^{n-1} (f)^{-i}(\mathcal{V}_0), [0,a]), N(\bigvee_{i=0}^{n-1} (g)^{-i}(\mathcal{V}_1), [a,1])\})}{n}
\]
\[
= \limsup_{n \to \infty} \frac{\log(\max\{N(\bigvee_{i=0}^{n-1} (f)^{-i}(\mathcal{V}_0), [0,a]), N(\bigvee_{i=0}^{n-1} (g)^{-i}(\mathcal{V}_1), [a,1])\}}{n}
\]
\[
= \max\{\text{Ent}(f, \mathcal{V}_0), \text{Ent}(g, \mathcal{V}_1)\}.
\]
Thus $\text{Ent}(h) \leq \max\{\text{Ent}(f), \text{Ent}(g)\}$. It follows from Lemma 9 that $\text{Ent}(h) = \max\{\text{Ent}(f), \text{Ent}(g)\}$. \qed

**Theorem 11.** Suppose that $\{a_n\}_{n=0}^{\infty}$ is a strictly increasing sequence in $[0,1)$ that converges at 1 where $a_0 = 0$. Let
\[
f_n : [a_n, a_{n+1}] \to [a_n, a_{n+1}],
\]
such that $f_n(a_{n+1}) = f_{n+1}(a_{n+1})$ for every $n \geq 0$. Define $f : [0,1] \to [0,1]$ by
\[
f(x) = \begin{cases} 
f_n(x) & \text{if } x \in [a_n, a_{n+1}] \\
1 & \text{if } x = 1,
\end{cases}
\]
Then $\text{Ent}(f) = \sup\{\text{Ent}(f_n)\}$.

Proof. By Lemma 10, it follows that
\[
\text{Ent}(f) = \max\{\text{Ent}(f|_{[0,a_n]}))_{i=0}^{n-1}, \text{Ent}(f|_{[a_{n+1}]})
\]
\[
= \max\{\text{Ent}(f_n))_{i=0}^{n-1}, \text{Ent}(f|_{[a_{n+1}])}
\]
\[
= \sup_n \{\text{Ent}(f_n)\}.
\]

Let $f$ be a continuous function on then interval $I$. Define the extension of $f$ by
\[
E(f)(x) = \begin{cases} 
1/2f(2x) & \text{if } x \in [0,1/2] \\
(2 - f(1))x + f(1) - 1 & \text{if } x \in (1/2, 1],
\end{cases}
\]
and the two extension of $f$ by
\[
E_2(f)(x) = \begin{cases} 
(f(0) + 1)x & \text{if } x \in [0, 1/3) \\
1/3f(3(x - 1/3)) + 1/3 & \text{if } x \in [1/3, 2/3] \\
(2 - f(1))x + f(1) - 1 & \text{if } x \in (2/3, 1].
\end{cases}
\]
Let \( \{g_i\}_{i=0}^{\infty} \) be such that the domain of \( g_i \) is the range of \( g_{i+1} \). Then define \( g_{i+1}^i = g_i \) and \( g_i^{i+k} = g_i \circ g_{i+1} \circ \ldots \circ g_{i+k-1} \).

**Theorem 12.** \( \text{Ent}(E_2(f)) = \text{Ent}(E(f)) = \text{Ent}(f) \).

**Proof.** Proof will be for \( \text{Ent}(E(f)) = \text{Ent}(f) \). Proof that \( \text{Ent}(E_2(f)) = \text{Ent}(f) \) is similar. Let \( a = E(f)(1/2) \) There are 2 Cases:

- **Case 1:** \( a < 1/2 \).
  
  First let \( a_0 = 1/2, a_1 = (E(f))^{-1}(a_0) \cap (1/2, 1] \) and then inductively define \( a_n = (E(f))^{-1}(a_{n-1}) \).

  (Note that \( E(f) \) is 1-1 on \( [a_1, 1] \)). Also notice that \( a_n \to 1 \) as \( n \) increases. Next let

  \[
  g : [0, 1/2] \to [0, 1/2],
  
  g_0 : [1/2, a_1] \to [a, 1/2],
  
  g_1 : [a_1, a_2] \to [1/2, a_1],
  
  \ldots
  
  
  g_k : [a_k, a_{k+1}] \to [a_{k-1}, a_k],
  
\]

  where \( g(x) = E(f)(x) \) and \( g_i(x) = E(f)|_{[a_i, a_{i+1}]}(x) \). Notice that for each \( i \geq 0, g_i \) is 1-1 and onto.

  Let \( \mathcal{V}_0 \) be an open cover of \( [0, 1/2] \) then define

  \[
  \mathcal{V}_1 = \{g_0^{-1}(V \cap [a, 1/2]) | V \in \mathcal{V}_0\} = \{(E(f))^{-1}(V) \cap [1/2, a_1] | V \in \mathcal{V}_0\}.
  
\]

  Continuing inductively define

  \[
  \mathcal{V}_k = \{g_k^{-1}(V) | V \in \mathcal{V}_{k-1}\} = \{(E(f))^{-k}(V) \cap [a_{k-1}, a_k] | V \in \mathcal{V}_0\}.
  
\]

  Clearly, \( \mathcal{V}_k \) is an open cover of \( [a_{k-1}, a_k] \).

  Next define

  \[
  W_n(\mathcal{V}_0) = \bigcup_{k=0}^{n} \mathcal{V}_k \cup \{(a_n, 1]\}.
  
\]
It will be shown that $\text{Ent}(E(f), W_n(\mathcal{V}_0)) = \text{Ent}(g, \mathcal{V}_0)$ for any $n \geq 0$.

First notice that

1. $\mathcal{V}_0$ is invariant under $g$
2. $g^{-1}_0 \mathcal{V}_0 \vee \mathcal{V}_1 = \mathcal{V}_1$
3. $g_i^{-1}(\mathcal{V}_i) = \mathcal{V}_{i+1}$ for $i \geq 1$
4. $(E(f))^{-1}(\{a_n, 1\}) = (a_{n+1}, 1]$,

$$W_n(\mathcal{V}_0) \vee g^{-1}(W_n(\mathcal{V}_0)) = (\mathcal{V}_0 \vee (E(f))^{-1}(\mathcal{V}_0)) \cup \bigcup_{k=1}^{n+1} \mathcal{V}_k \cup \{a_{n+1}, 1\}.$$ 

Next let

$$\mathcal{Q}_k = \{U \cap [0, 1/2] | U \in \bigvee_{i=0}^k g^{-i}(\mathcal{V}_0)\}$$

and

$$\mathcal{Q}_k' = \{U \cap [a, 1/2] | U \in \bigvee_{i=0}^k g^{-i}(\mathcal{V}_0)\}.$$

Then it follows that for $m < n$

$$\bigvee_{i=0}^m (E(f))^{-i}(W_n(\mathcal{V}_0)) = \mathcal{Q}_m \cup \bigcup_{i=1}^m (g_0^i)^{-1}(\mathcal{Q}_{m-i}) \cup \bigcup_{i=m}^{m+n} \mathcal{V}_i \cup \{a_{n+m+1}, 1\},$$

and for $m \geq n$

$$\bigvee_{i=0}^m (E(f))^{-i}(W_n(\mathcal{V}_0)) = \mathcal{Q}_m \cup \bigcup_{i=1}^n (g_0^i)^{-1}(\mathcal{Q}_{m-i}) \cup \bigcup_{i=m}^{m+n} \mathcal{V}_i \cup \{a_{n+m+1}, 1\}.$$ 

So for $m \geq n$

$$N(\bigvee_{i=0}^m (E(f))^{-i}(W_n(\mathcal{V}_0)), [0, 1]) = N(\bigvee_{i=0}^m \mathcal{Q}_m, [0, 1/2]) + \sum_{i=1}^n N((g_0^i)^{-1}(\mathcal{Q}_{m-i}), [a_{i-1}, a_i]) + \sum_{i=m}^{n+m} N(\mathcal{V}_i, [a_{i-1}, a_i]) + 1$$

$$\leq (n + m + 1)N(\bigvee_{i=0}^m \mathcal{Q}_m, [0, 1/2]) + 1.$$ 

Hence,

$$\text{Ent}(E(f), W_n(\mathcal{V}_0)) = \lim_{m \to \infty} \frac{\log N(\bigvee_{i=0}^m (E(f))^{-i}(W), [0, 1])}{m + 1}$$

$$\leq \lim_{m \to \infty} \frac{\log((n + m + 1)N(\bigvee_{i=0}^m \mathcal{Q}_m, [0, 1/2]) + 1)}{m + 1}$$

$$= \lim_{m \to \infty} \frac{\log N(\bigvee_{i=0}^m \mathcal{Q}_m, [0, 1/2])}{m + 1}$$

$$= \text{Ent}(f, \mathcal{V}_0).$$

Now let $\mathcal{U}$ be any finite open cover of $[0, 1]$. Then there exists a finite open cover $\mathcal{V}$ of $[0, 1/2]$ with sufficiently small mesh and an integer $n$ sufficiently large such that $W_n(\mathcal{V})$ refines $\mathcal{U}$. Hence

$$\text{Ent}(E(f), \mathcal{U}) \leq \text{Ent}(E(f), W_n(\mathcal{V})) \leq \text{Ent}(f, \mathcal{V}).$$
Thus $\text{Ent}(E(f)) \leq \text{Ent}(f)$. Since both $2x$ and $(1/2)x$ are homeomorphisms, it follows from Corollary 5 that $\text{Ent}(f) = \text{Ent}(1/2f(2x))$. Also from Lemma 9 it follows that $\text{Ent}(1/2f(2x)) \leq \text{Ent}(E(f))$. Hence, $\text{Ent}(E(f)) = \text{Ent}(f)$.

**Case 2** $a = 1/2$.

Then $E(f)$ is the identity on $[1/2, 1]$. Thus it also follows from Lemma 9 and Corollary 5 that $\text{Ent}(E(f)) = \text{Ent}(f)$.

Define the **halving** of $f$ by

$$H(f)(x) = \begin{cases} 1 - 1/2E(f)(2x) & \text{if } x \in [0, 1/2] \\ 1 - x & \text{if } x \in [1/2, 1] \end{cases},$$

and the **doubling** $f$ by

$$D(f)(x) = \begin{cases} 1/2E(f)(2x) & \text{if } x \in [0, 1/2] \\ 1 - 1/2E(f)(2 - 2x) & \text{if } x \in [1/2, 1] \end{cases}.$$

(See Figure 2.)

Define $H^2(f) = H(H(f))$ and continuing inductively define $H^n(f) = H(H^{n-1}(f))$. On the other hand $(H(f))^2 = H(f) \circ H(f)$. Notice that $(H(f))^2 = D(f)$.

**Theorem 13.** $\text{Ent}(D(f)) = \text{Ent}(f)$.

**Proof.** Notice that $h_1(x) = (1/2)x$ and $h_2(x) = 1 - (1/2)x$ are homeomorphisms. Thus by Corollary 5,

$$\text{Ent}(E(f)) = \text{Ent}(h_1 \circ E(f) \circ h_1^{-1}) = \text{Ent}(1/2E(f)(2x))$$

and

$$\text{Ent}(E(f)) = \text{Ent}(h_2 \circ E(f) \circ h_2^{-1}) = \text{Ent}(1 - 1/2E(f)(2 - 2x)).$$

So by Lemma 10 and Theorem 12,

$$\text{Ent}(D(f)) = \max\{\text{Ent}(1/2E(f)(2x)), \text{Ent}(1 - 1/2E(f)(2 - 2x))\}$$

$$= \text{Ent}(E(f))$$

$$= \text{Ent}(f).$$

**Figure 2.** $D(T)(x)$ and $H(T)(x)$
Theorem 14. Ent($H^n(f)$) = $\frac{1}{2^n}$ Ent($f$).

Proof. First, since $(H(f))^2 = D(f)$, it follows by Proposition 2 and Theorem 13 that

$$\text{Ent}(H(f)) = \frac{1}{2} \text{Ent}(D(f)) = \frac{1}{2} \text{Ent}(f).$$

Continuing inductively suppose that Ent($H^{n-1}(f)$) = $\frac{1}{2^{n-1}}$ Ent($f$). Then it follows that

$$\text{Ent}(H^n(f)) = \text{Ent}(H(H^{n-1}(f)))$$
$$= \frac{1}{2} \text{Ent}(H^{n-1}(f))$$
$$= \frac{1}{2^n} \text{Ent}(f).$$

$\square$

The next theorem is the main result of this paper:

Theorem 15. There exists an arc-like continuum $X$ such that for every $\epsilon \in [0, \infty]$ there exists a homomorphism $g_\epsilon : X \to X$ such that Ent($g_\epsilon$) = $\epsilon$. Furthermore, $X$ does not contain a pseudo-arc.

Proof. Now for the construction of the arc-like continuum. For each integer $n \geq 1$ let

$$s_n : \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}\right] \to \left[0, \frac{2^n - 1}{2^n}\right]$$

be defined by

$$s_n(x) = \frac{2^n - 1}{2^n} + \frac{1}{2^n} E_2(H^n(T))(2^n(x - \frac{2^n - 1}{2^n})).$$

Then by Theorems 3, 12 and 14, Ent($s_n$) = $\frac{1}{2^n}$ log(2). Next define

$$g(x) = \begin{cases} s_n(x) & \text{if } x \in \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}\right] \text{ for } n = 0, 1, 2, ... \\
1 & \text{if } x = 1. \end{cases}$$

Let $X = \lim \left\{\{0, 1\}, g\right\}_{i=1}^\infty$ and $X_n = \lim \left\{\left\{\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}, s_n\right\}_{i=1}^\infty \right\}$ for $n = 0, 1, 2, ...$ Then since $s_n$ is composed of a finite number of monotone pieces, each $X_n$ does not contain a pseudo-arc by Corollary 8. Thus, $X = \left\{\{0, 1\}, \bigcup_{n=0}^\infty X_n\right\}$ does not contain a pseudo-arc (see Figure 3).

Given $\epsilon > 0$, there exist positive integers $c_n \in \mathbb{N}$ such that $\frac{c_n}{2^n} \log(2) \leq \epsilon$ and $\lim_{n \to \infty} \frac{c_n}{2^n} \log(2) = \epsilon$. For $\epsilon = 0$ let $c_n = 0$ and for $\epsilon = \infty$ let $c_n = 2^n$. Then take

$$g_\epsilon(x) = \begin{cases} s_n(x) & \text{if } x \in \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}\right] \text{ for } n = 0, 1, 2, ... \\
1 & \text{if } x = 1. \end{cases}$$

So by Proposition 2, Ent($s_n^\epsilon(x)$) = $\frac{c_n}{2^n}$ log(2). Thus, it follows from Theorem 11 that Ent($g_\epsilon$) = $\epsilon$. Also since $s_n \circ s_n^\epsilon = s_n^\epsilon \circ s_n$, it follows that $g \circ g_\epsilon = g_\epsilon \circ g$. Let $\widehat{g}_\epsilon : X \to X$ be defined by $\widehat{g}_\epsilon((x_i)) = \langle g_\epsilon(x_i) \rangle$. Then by Theorem 6, Ent($\widehat{g}_\epsilon$) = Ent($g_\epsilon$) = $\epsilon$. Also, notice that $\widehat{g}_\epsilon\left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}}\right] = s_n^\epsilon$ which is a shift homeomorphism on $X_n$. Thus, it follows that $\widehat{g}_\epsilon$ is a homeomorphism. $\square$
Question 1. If $h : P \to P$ is a homeomorphism on the pseudo-arc $P$, must $\text{Ent}(P) = 0$ or $\text{Ent}(P) = \infty$?

And if so, then consider the following question:

Question 2. If $h : X \to X$ is a homeomorphism on a hereditarily indecomposable continuum $X$, must $\text{Ent}(h) = 0$ or $\text{Ent}(h) = \infty$?

4. References


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