Positive entropy on nonautonomous interval maps and the topology of the inverse limit space

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Accepted 27 September 2006

Abstract

Entropy on nonautonomous maps \( \{f_i\}_{i=0}^{\infty} \) of the interval is defined 2 ways. Under one definition, called forward entropy, it is shown that positive entropy implies that the inverse limit space of \((\{f_i\}_{i=0}^{\infty}, I)\) contains an indecomposable subcontinuum. Under the second definition, called backwards entropy, it is shown that the inverse limit space of \((\{f_i\}_{i=0}^{\infty}, I)\) is not locally connected.

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MSC: primary 54H20, 54F50; secondary 54E40

Keywords: Entropy; Inverse limit; Indecomposable continuum

1. Introduction

There is much known about the relationship between the entropy of onto maps of the interval \( f: I \to I \) and the inverse limit space on \( f \). Primarily, it was shown by Barge and Martin [2] that if the entropy of \( f \) is positive, then the inverse limit space must contain a nondegenerate indecomposable subcontinuum. On the other hand, if we consider an infinite collection of onto nonautonomous maps on the interval, not much is known about the topology of the inverse limit space. In this paper, entropy is defined two different ways on an infinite collection of maps \( \{f_i\}_{i=0}^{\infty} \). Both definitions are equivalent to the original definition of entropy if applied to the same map \( f \). Then under an additional hypothesis of shift-invariance, which again is always satisfied by entropy of just one map, it is shown that under one definition positive entropy implies a nonlocally connected inverse limit and under the other definition positive entropy implies an indecomposable subcontinuum in the inverse limit.

A map is a continuous function. A continuum \( X \) is decomposable provided there exist proper subcontinua \( H \) and \( K \) such that \( X = H \cup K \). A continuum is indecomposable if it is not decomposable. If \( \mathcal{U} \) is a collection of open sets, the mesh of \( \mathcal{U} \) is defined by mesh \( (\mathcal{U}) = \sup \{\text{diam}(U): U \in \mathcal{U}\} \). A chain \( \mathcal{C} \) is an indexed collection of open sets \( \{C_1, C_2, \ldots, C_n\} \) such that \( C_i \cap C_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). \( \mathcal{C} = \{C_1, C_2, \ldots, C_n\} \) is a proper chain if \( \overline{C}_i \cap \overline{C}_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). A continuum \( X \) is chainable if for every \( \epsilon > 0 \) there exists a chain covering \( X \) such that mesh \( (\mathcal{C}) < \epsilon \). Every chainable continuum can be covered by a proper chain with arbitrarily small mesh. Likewise, if \( X \) is the inverse limit of arcs, then \( X \) is also chainable. Chainable continua are also called arc-like and

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snake-like continua. A collection of sets $\mathcal{V}$ refines a collection $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$.

The original definition of entropy is due to Adler et al. [1]. Let $X$ be a compact metric space and $f : X \to X$ be a continuous function. If $\mathcal{U}$ is a finite open cover of $X$, then define $N(\mathcal{U})$ to be the number of sets in a finite subcover of $\mathcal{U}$ with smallest cardinality. If $\mathcal{U}$ and $\mathcal{V}$ are two open covers of $X$, let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$. Also, define

$$
\forall \mathcal{U} \in \mathcal{V} \ni \frac{1}{n-1} \mathcal{U} = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-(n+1)}(\mathcal{U}), \quad \text{where } f^0 = \text{id}
$$

and

$$
\text{Ent}(f, \mathcal{U}) = \lim_{n \to \infty} (1/n) \log N\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})\right).
$$

Then the topological entropy of $f$ is defined as

$$
\text{Ent}(f) = \sup \{\text{Ent}(f, \mathcal{U}) : \mathcal{U} \text{ is an open cover of } X\}. 
$$

In the above definition, if we take $\mathcal{A}$ to be a finite closed cover then

$$
\overline{\text{Ent}}(f) = \sup \{\text{Ent}(f, \mathcal{A}) : \mathcal{A} \text{ is an finite closed cover of } X\}. 
$$

Next, the first definition of entropy on a collection of maps will be called backwards entropy and is due to Kolyada and Snoha [3]. Suppose that $\{f_n\}_{n=0}^{\infty}$ is a collection of continuous surjections $f_n : X \to X$ and $\mathcal{U}$ is a finite open cover of $X$. Let $f_n^i = f_i \circ f_{i+1} \circ \cdots \circ f_{n-1}$ where $f_i^0 = \text{id}$ and $f_i^{i+1} = f_i$. Define

$$
\text{Ent}_B(\{f_n\}_{n=0}^{\infty}, \mathcal{U}) = \limsup_{n \to \infty} (1/n) \log N\left(\bigvee_{i=0}^{n-1} (f_i^0)^{-1}(\mathcal{U})\right).
$$

Then the topological entropy of $\{f_n\}_{n=0}^{\infty}$ is defined as

$$
\text{Ent}_B(\{f_n\}_{n=0}^{\infty}) = \sup \{\text{Ent}_B(\{f_n\}_{n=0}^{\infty}, \mathcal{U}) : \mathcal{U} \text{ is an finite open cover of } X\}. 
$$

On the other hand, define forward entropy in the following way:

$$
\text{Ent}_F(\{f_n\}_{n=0}^{\infty}, X, \mathcal{U}) = \limsup_{n \to \infty} (1/n) \log N\left(\bigvee_{i=0}^{n-1} (f_i^{n-1})^{-1}(\mathcal{U})\right).
$$

Then define $\text{Ent}_F(\{f_n\}_{n=0}^{\infty}, \overline{\text{Ent}}_B(\{f_n\}_{n=0}^{\infty})$ and $\overline{\text{Ent}}_F(\{f_n\}_{n=0}^{\infty})$ in similar ways.

Notice that both of these definitions agree with the original definition of entropy on a single map $f_i = f$ since

$$
\bigvee_{i=0}^{n-1} (f_i^{n-1})^{-1}(\mathcal{U}) = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U}) = \bigvee_{i=0}^{n-1} (f_i^0)^{-1}(\mathcal{U}).
$$

**Proposition 1.** Suppose that $\mathcal{U}, \mathcal{V}$ are covers of $X$ such that $\mathcal{V}$ refines $\mathcal{U}$. Then

$$
\text{Ent}_B(\{f_n\}_{n=0}^{\infty}, \mathcal{U}) \leq \text{Ent}_B(\{f_n\}_{n=0}^{\infty}, \mathcal{V})
$$

and

$$
\text{Ent}_F(\{f_n\}_{n=0}^{\infty}, \mathcal{U}) \leq \text{Ent}_F(\{f_n\}_{n=0}^{\infty}, \mathcal{V}).
$$

**Proof.** Proof will be for backwards entropy. The proof for forwards entropy will be similar. Let $Q_\mathcal{V}$ be a minimal subcover of $\bigvee_{i=0}^{n-1} (f_i^{n-1})^{-1}(\mathcal{V})$. Then for each $V_i$ such that $Q_\mathcal{V} = \bigcap_{i=0}^{n-1} (f_i^{n-1})^{-1}(V_i) \in Q_\mathcal{V}$ there exists $U_i \in \mathcal{U}$ such that $V_i \subset U_i$. Let $Q_\mathcal{U} = \bigcap_{i=0}^{n-1} (f_i^{n-1})^{-1}(U_i)$ and $\mathcal{Q}_\mathcal{U}$ be the collection of the so defined $Q_\mathcal{U}$’s. Then

$$
N\left(\bigvee_{i=0}^{n-1} (f_i^{n-1})^{-1}(\mathcal{V})\right) = |Q_\mathcal{V}| \geq |Q_\mathcal{U}| \geq N\left(\bigvee_{i=0}^{n-1} (f_i^{n-1})^{-1}(\mathcal{U})\right).
$$

Then using the definition of entropy, the proposition follows. \( \square \)
2. Entropy of subsets of covers

Let $S$ be a collection of symbols. A word on $S$ is a finite or infinite sequence of symbols in $S$. If $A$ is a cover (open or closed) of $I$, we can find the entropy of $B \subset A$ with respect to a collection of maps $\{f_i\}_{i=0}^\infty$. To do this, we look at entropy as a growing collection of words defined in the following way: Let $A$ be a finite cover of $I$, $A_i \in A$ and $\langle A_i \rangle_{i=0}^\infty \in \mathcal{W}^B(\{f_i\}_{i=0}^\infty, A) \ (\mathcal{W}^B \text{ when there is no confusion})$ provided

$$\text{int}(A_0 \cap (f_1^{-1}(A_1) \cap \cdots \cap (f_i^{-1}(A_i))) \neq \emptyset$$

for each $i$. Likewise, define $\langle A_i \rangle_{i=0}^\infty \in \mathcal{W}^F(\{f_i\}_{i=0}^\infty, A) \ (\mathcal{W}^F \text{ when there is no confusion})$ provided

$$\text{int}((f_0^{-1}(A_0) \cap (f_1^{-1}(A_1) \cap \cdots \cap (f_i^{-1}(A_i))) \neq \emptyset$$

for each $i$. $\mathcal{W}$ will be used when it does not matter if the words are of the form $\mathcal{W}^B$ or $\mathcal{W}^F$. Define the prefix of length $k$ of $\langle A_i \rangle_{i=0}^\infty$ (or $\langle A_i \rangle_{i=0}^n$ where $n > k$) to be $\Pi_k(\langle A_i \rangle_{i=0}^\infty) = \langle A_i \rangle_{i=0}^k$ $\Pi_k(W) = \{\langle A_i \rangle_{i=0}^k | \langle A_i \rangle_{i=0}^\infty \in \mathcal{W}\}$ and $\pi_k(\langle A_i \rangle_{i=0}^\infty) = A_k$. Then define the entropy of a collection of infinite words by

$$\text{ENT}(\mathcal{W}) = \limsup_{k \to \infty} \frac{\log(|\Pi_k(\mathcal{W})|)}{k}.$$ 

Notice that

$$\text{Ent}_B(\{f_i\}_{i=0}^\infty, A) \leq \limsup_{k \to \infty} \frac{\log|\Pi_k(\mathcal{W}^B)|}{k} = \text{ENT}(\mathcal{W}^B)$$

and

$$\text{Ent}_F(\{f_i\}_{i=0}^\infty, A) \leq \limsup_{k \to \infty} \frac{\log|\Pi_k(\mathcal{W}^F)|}{k} = \text{ENT}(\mathcal{W}^F).$$

Also notice that if $A$ is a partition of $I$, that is of the form $\{(a_i, a_{i+1}]\}_{i=0}^n$ where $0 = a_0$ and $1 = a_n$, then the above inequalities become equalities.

Next, suppose that $B \subset A$. We can define the entropy (both forwards and backwards) of $\{f_i\}_{i=0}^\infty$ on $B$ in the following way: Let $G = \{g(i)\}_{i=1}^\infty$ be a sequence of increasing positive integers. If $W \in \mathcal{W}(\{f_i\}_{i=0}^\infty, A)$, let $\Pi_G(W) = \{\Pi_i(W) \}_{i=1}^{\infty}$ and $\mathcal{W}(\{f_i\}_{i=0}^\infty, A, G) = \{\Pi_G(W) | W \in \mathcal{W}\} \ (\mathcal{W}_G \text{ when there is no confusion}).$ Next define

$$\mathcal{W}(\{f_i\}_{i=0}^\infty, B, G) = \{W \in \mathcal{W}_G | \pi_i(W) \in B \text{ for } i \geq 0\}.$$ 

Then the entropy of $\{f_i\}_{i=0}^\infty$ on $(B, G)$ is

$$\text{Ent}(\{f_i\}_{i=0}^\infty, B, G) = \limsup_{i \to \infty} \frac{\log(|\Pi_i(\mathcal{W}(\{f_i\}_{i=0}^\infty, B, G))|)}{g(i)}$$

and the entropy of $\{f_i\}_{i=0}^\infty$ on $B$ is

$$\text{Ent}(\{f_i\}_{i=0}^\infty, B) = \sup_{G} \text{Ent}(\{f_i\}_{i=0}^\infty, B, G).$$

Notice that $\text{Ent}(\{f_i\}_{i=0}^\infty, A) \geq \text{Ent}(\{f_i\}_{i=0}^\infty, B)$ and since $g(i) \geq i$ then

$$|\Pi_{g(i)}(\mathcal{W}(\{f_i\}_{i=0}^\infty, B, G))| \geq |\Pi_i(\mathcal{W}(\{f_i\}_{i=0}^\infty, B, G))|,$$

and it follows that

$$\text{ENT}(\mathcal{W}(\{f_i\}_{i=0}^\infty, B, G)) \geq \text{Ent}(\{f_i\}_{i=0}^\infty, B, G).$$

The next theorem can be found in [4] and is true for both forward and backward entropy.

**Theorem 2.** If $\text{Ent}(\{f_i\}_{i=0}^\infty, A) > 0$ then there exist elements $A, B \in A$ such that $\text{Ent}(\{f_i\}_{i=0}^\infty, \{A, B\}) > 0.$
3. Inequivalence between open and closed entropy

In this section we will see how open and closed entropy can differ on infinite collections of maps.

Proposition 3.

\[
\overline{\text{Ent}}_B \left( \{ f_n \}_{n=0}^{\infty} \right) \geq \text{Ent}_B \left( \{ f_n \}_{n=0}^{\infty} \right)
\]

and

\[
\overline{\text{Ent}}_F \left( \{ f_n \}_{n=0}^{\infty} \right) \geq \text{Ent}_F \left( \{ f_n \}_{n=0}^{\infty} \right).
\]

Proof. Let \( U \) be any finite open cover of \( X \) and \( A_U \) be a finite closed cover that refines \( U \). Then for each \( n \),

\[
\bigvee_{i=0}^{n-1} \left( f_i \right)^{-1} \left( A_U \right) \text{ refines } \bigvee_{i=0}^{n-1} \left( f_i \right)^{-1} \left( U \right)
\]

and

\[
N \left( \bigvee_{i=0}^{n-1} \left( f_i \right)^{-1} \left( A_U \right) \right) \geq N \left( \bigvee_{i=0}^{n-1} \left( f_i \right)^{-1} \left( U \right) \right)
\]

and the proposition follows. \( \square \)

Example 4. The following function is called the trifold:

\[
f(x) = \begin{cases} 
3x & x \in [0, \frac{1}{2}], \\
2 - 3x & x \in (\frac{1}{2}, \frac{3}{2}), \\
3x - 2 & x \in [\frac{3}{2}, 1]. 
\end{cases}
\]

(See Fig. 1.)

Let \( A = \{ [0, 1/2], [1/2, 1] \} \). Then since \( f([0, 1/2]) = [0, 1] \) and \( f([1/2, 1]) = [0, 1] \), it is easy to see that \( N \left( \bigvee_{i=0}^{n-1} f^{-i} (A) \right) = 2^n \). Hence, \( \text{Ent}(f) > 0 \).

The next example shows that the inequalities in Proposition 3 are not necessarily equalities.

Example 5. Let \( g_0 = f \) and for \( n \geq 0 \) and \( 2^n \leq i < 2^{n+1} \) define \( A_0 = \frac{1}{2}, A_n = A_{n-1} + \frac{1}{2^{2n+1}} \) and \( g_i : I \to I \) by

\[
g_i(x) = \begin{cases} 
x & x \in [0, \frac{1}{2} - \frac{A_n}{2}] \cup [\frac{1}{2} + \frac{A_n}{2}, 1], \\
3x + A_n - 1 & x \in (\frac{1}{2} - \frac{A_n}{2}, \frac{1}{2} - \frac{A_n}{6}], \\
2 - 3x & x \in (\frac{1}{2} - \frac{A_n}{6}, \frac{1}{2} + \frac{A_n}{6}], \\
3x - A_n - 1 & x \in (\frac{1}{2} + \frac{A_n}{6}, \frac{1}{2} + \frac{A_n}{2}). 
\end{cases}
\]

(See Fig. 2.)
Proposition 6. Let \( \{g_i\}_{i=0}^{\infty} \) be defined as in Example 5. Then \( \overline{\text{Ent}}_B(\{g_i\}_{i=0}^{\infty}, X) > 0 \) and \( \overline{\text{Ent}}_F(\{g_i\}_{i=0}^{\infty}, X) > 0 \).

Proof. Let \( A = \{A, B\} \) where \( A = [0, 1/2] \) and \( B = [1/2, 1] \). For \( n \geq 1 \), it follows that \([1/2 - A_n/2, 1] \subset (g_0^{2^n})^{-1}(A)\) and \([1/2, 1 + A_n/2] \subset (g_0^{2^n})^{-1}(B)\). Since \([1/2 - A_n/2, 1/2 + A_n/2]\) is invariant in \( g_i \) for \( 2^n \leq i < 2^{n+1} \), it follows in a similar manner as Example 4 that,

\[
N\left(\frac{1}{i=0} (g_0^{2^n+i})^{-1}(A)\right) = N\left(\frac{1}{i=2^n} (g_i^{2^n+1})^{-1}(A)\right) = N\left(\frac{1}{i=0} (g_2^{2^n})^{-1}(A)\right) = 2^n.
\]

Hence, \( N(\frac{1}{i=0} (g_0^{i})^{-1}(A)) \geq 2^n \) and \( N(\frac{1}{i=0} (g_2^{2^n})^{-1}(A)) \geq 2^n \). So we may conclude that

\[
\overline{\text{Ent}}_B(\{g_i\}_{i=1}^{\infty}, X) \geq (1/2) \log 2
\]

and

\[
\overline{\text{Ent}}_F(\{g_i\}_{i=1}^{\infty}, X) \geq (1/2) \log 2. \quad \square
\]

However, note that the inverse limit of \( \{g_i\}_{i=0}^{\infty} \) is homeomorphic to an arc.

If \( \mathcal{U} \) is a collection of sets, define \( \mathcal{U}^* = \bigcup_{U \in \mathcal{U}} U \). For \( U \in \mathcal{U} \), define \( \text{core}(U) = U - (U - \{U\})^c \).

Proposition 7. Let \( \{g_i\}_{i=0}^{\infty} \) be defined as in Example 5. Then \( \text{Ent}_B(\{g_i\}_{i=0}^{\infty}, X) = 0 \) and \( \text{Ent}_F(\{g_i\}_{i=0}^{\infty}, X) = 0 \).

Proof. Let \( \mathcal{U} \) be any open cover of \( I \). Then there exists a chain cover \( \mathcal{V} \) that refines \( \mathcal{U} \) and a \( V \in \mathcal{V} \) such that \( 1/2 \in \text{core}(V) \). Let \( \epsilon \) be such that \((1/2 - \epsilon, 1/2 + \epsilon) \subset \text{core}(V) \). However, there exists an integer \( N \) such that \( \epsilon > \frac{1}{3^N} \). Hence,

\[
\frac{1}{3^i} \epsilon > \frac{1}{3^{i+N}} > \frac{1}{3^{2^n+1+N}} \geq \frac{1}{3^{2^n+1}} A_n - 1 = A_n
\]

for every \( 2^n \leq i < 2^{n+1} \) and whenever \( n > N + 1 \). It follows that

\[
\left(1/2 - A_n/2, 1/2 + A_n/2\right) \subset (g_0^i)^{-1}(\text{core}(V)).
\]

Therefore, \((g_0^i)^{-1}(V) = (g_0^{2^n})^{-1}(V)\) and \((g_2^i)^{-1}(V) = (g_2^{2^n})^{-1}(V)\) for every \( i \geq 2^N \). Hence, with an application of Proposition 1, we have

\[
\text{Ent}_B(\{g_i\}_{i=1}^{\infty}, \mathcal{U}) \leq \text{Ent}_B(\{g_i\}_{i=1}^{\infty}, \mathcal{V})
\]

\[
= \limsup_{k \to \infty} \frac{\log(|N(\bigvee_{i=0}^k (g_0^i)^{-1}(V))|)}{k}
\]

\[
= \limsup_{k \to \infty} \frac{\log(|N(\bigvee_{i=0}^{2^n} (g_0^i)^{-1}(V))|)}{k} = 0
\]
Hence, there must be at least \( \Pi_k(\{\mathcal{A}_i\}_{i=1}^\infty, \mathcal{V}) \) and \( \Pi_k(\{\mathcal{B}_i\}_{i=1}^\infty, \mathcal{V}) \). Since \( \mathcal{U} \) was an arbitrary open cover, we may conclude that the entropies are 0. \( \Box \)

However, there are instances when closed positive entropy implies open positive entropy.

**Theorem 8.** Suppose that there exist disjoint closed intervals \( A, B \) such that \( \text{Ent}(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}) > 0 \). Then there exists a finite open cover \( \mathcal{U} \) such that \( \text{Ent}(\{\mathcal{A}_i\}_{i=0}^\infty, \mathcal{U}) > 0. \)

**Proof.** Proof will be for backward entropy. Proof for forward entropy is similar. Since \( A \) and \( B \) are disjoint closed sets, there exist open sets \( V_A, V_B, U_A, U_B \) such that

\[
A \subset V_A \subset \overline{V}_A \subset U_A, \quad B \subset V_B \subset \overline{V}_B \subset U_B
\]

and \( \overline{U}_A \cap \overline{U}_B \neq \emptyset \). Let \( U_R = I - \{\overline{V}_A \cup \overline{V}_B\} \). Then \( \mathcal{U} = \{U_A, U_B, U_R\} \) is an open cover of \( I \).

Next, there exists a \( G = \{g(i)\}_{i=1}^\infty \) such that \( \text{Ent}_B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G) > 0 \). So for every \( W \in \Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G)) \) there exists a \( x_W \in \text{int}(\bigcap_{i=1}^k f_0^{g(i)}(A_i)) \) where \( A_i \in \{A, B\} \). Then every \( V = \bigcap_{i=0}^k (f_0^{g(i)})^{-1}(U_i) \) such that \( U_i \in \mathcal{U} \) and \( x_W \in V \) must have the property that

1. \( U_i = U_A \) if and only if \( \pi_i(W) = A \).
2. \( U_i = U_B \) if and only if \( \pi_i(W) = B \).

Hence, there must be at least \( |\Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G))| \) elements of \( \bigcap_{i=0}^k (f_0^{g(i)})^{-1}(U) \) to cover \( \{x_W | W \in \Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G))\} \). It follows that

\[
|\Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G))| \leq N\left(\sqrt[k]{(f_0^{g(i)})^{-1}(\mathcal{U})}\right)
\]

and therefore

\[
\text{Ent}(\{\mathcal{A}_i\}_{i=0}^\infty, \mathcal{U}) = \limsup_{k \to \infty} \frac{\log(|\Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G))|)}{g(k)} \geq \limsup_{k \to \infty} \frac{\log(|\Pi_k(\mathcal{V}^B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G))|)}{g(k)} \geq \text{Ent}_B(\{\mathcal{A}_i\}_{i=0}^\infty, \{A, B\}, G).
\]

\( > 0. \) \( \Box \)

4. **Collections of words that are invariant under shifts**

We say that \( \mathcal{W} \) is \textit{invariant under shifts} provided that if \( \langle A_i \rangle_{i=0}^\infty \in \mathcal{W} \) then \( \sigma^k(\langle A_i \rangle_{i=0}^\infty) = \langle A_i \rangle_{i=k}^\infty \in \mathcal{W} \) for each \( k \geq 0 \). Here, \( \sigma \) is call the shift of \( \mathcal{W} \). If we use the same bonding map \( f \) then both \( \mathcal{W}^B(f, \mathcal{A}) \) and \( \mathcal{W}^F(f, \mathcal{A}) \) are always closed under shifts. It is this property that will give us the theorems on the interesting topology of the inverse limit space.
Example 9. Let \( f_0 = f \) as in Example 4 and define inductively

\[
 f_i(x) = \begin{cases}
  \frac{1}{3} f_{i-1}(3x) & x \in [0, \frac{1}{3}), \\
  \frac{1}{2} + \frac{1}{3} f_{i-1}(3(x - \frac{1}{3})) & x \in [\frac{1}{3}, \frac{2}{3}), \\
  \frac{2}{3} + \frac{1}{3} f_{i-1}(3(x - \frac{2}{3})) & x \in (\frac{2}{3}, 1].
\end{cases}
\]

(See Fig. 3.)

Proposition 10. Let \( \{f_i\}_{i=0}^\infty \) be defined as in Example 9 and \( \mathcal{A} = ([0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{1}{3}, 1]) \). Then \( \mathcal{W}^B(\{f_i\}_{i=0}^\infty, \mathcal{A}) \) is closed under shifts and \( \text{Ent}_B(\{f_i\}_{i=0}^\infty, \mathcal{A}) = \log(3) \).

Proof. Notice that with the trifold \( f \) each of \( A_j \cap f^{-1}(A_j) \) is an arc of the form \([\frac{n}{3k}, \frac{n+1}{3k}]\) where \( n \in \{0, \ldots, 8\} \) and \( A_i, A_j \in \mathcal{A} \). In a similar way, each map \( f_k \) has an invariant trifold on each subarc of the form \([\frac{n}{3k}, \frac{n+1}{3k}]\) where \( n \in \{0, \ldots, 3^k - 1\} \). Let \( \mathcal{A}(n, k) = ([\frac{n}{3^k}, \frac{n+1}{3^k}), [\frac{3n+1}{3^{k+1}}), [\frac{3n+2}{3^{k+1}}), [\frac{3n+3}{3^{k+1}}), [\frac{n+1}{3^k}) \) (again \( n \in \{0, \ldots, 3^k - 1\} \)). Since \( f_2 \) is monotone on each element of \( \mathcal{A}(n, k) \), it follows that each \( \hat{A}_i \cap f_k^{-1}(A_j) \) is an arc of the form \([\frac{m}{3^{k+2}}, \frac{m+1}{3^{k+2}}]\) where \( m \in \{9n, \ldots, 9n + 8\} \) and \( \hat{A}_i, \hat{A}_j \in \mathcal{A}(n, k) \). Recall that \( f_i^n = f_i \circ f_i + 1 \circ \cdots \circ f_i - n \). It follows inductively that if \( A_0 \cap (f_0^{-1})(A_1) \cap \cdots \cap (f_0^{-1})(A_{k-1}) \) is an arc of the form \([\frac{n}{3^{k+1}}, \frac{n+1}{3^{k+1}}]\), then

\[
 \mathcal{W}^B(\{f_i\}_{i=0}^\infty, \mathcal{A}) = \{ (A_i)_{i=0}^\infty \mid A_i \in \mathcal{A} \},
\]

which is clearly invariant under shifts, and that \( |\prod_k (\mathcal{W}^B(\{f_i\}_{i=0}^\infty, \mathcal{A}))| = 3^k \). Thus, \( \text{Ent}_B(\{f_i\}_{i=0}^\infty, \mathcal{A}) = \log(3) \).

Proposition 11. Let \( \{f_i\}_{i=0}^\infty \) be defined as in Example 9 and \( \mathcal{A} = ([0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{1}{3}, 1]) \) Then \( \mathcal{W}^F(\{f_i\}_{i=0}^\infty, \mathcal{A}) \) is invariant under shifts and \( \text{Ent}_F(\{f_i\}_{i=0}^\infty, \mathcal{A}) = 0 \).

Proof. Let \( A = [0, \frac{1}{3}), B = [\frac{1}{3}, \frac{2}{3}), \) and \( C = [\frac{2}{3}, 1] \). Since each of \( A, B, C \) is invariant under \( f_i^{-1} \) where \( i \geq 1 \), \( (f_i^k)^{-1}(A) = A \) for \( 0 < i \leq k \). Hence,

\[
 (f_0^k)^{-1}(A) \lor (f_0^k)^{-1}(A) \lor \cdots \lor (f_0^k)^{-1}(A) = (f_0^k)^{-1}(A) \lor A.
\]

Thus all of the elements of \( \mathcal{W}^F(\{f_i\}_{i=0}^\infty, \mathcal{A}) \) is of the form \( \langle Z, A, A, \ldots \rangle, \langle Z, B, B, \ldots \rangle \) or \( \langle Z, C, C, \ldots \rangle \) where \( Z \in \{A, B, C\} \). This is clearly closed since \( \langle A_i \rangle_{i=0}^\infty, \langle B \rangle_{i=0}^\infty \) and \( \langle C \rangle_{i=0}^\infty \in \mathcal{W}^F(\{f_i\}_{i=0}^\infty, \mathcal{A}) \). Furthermore, \( |\prod_k (\mathcal{W}_F(\{f_i\}_{i=0}^\infty, \mathcal{A}))| = 9 \) for all \( k > 0 \). Hence, \( \text{Ent}_F(\{f_i\}_{i=0}^\infty, \mathcal{A}) = 0 \).
Propositions 10 and 11 give a hint that forward and backward entropy are different. This will be evident in the next two sections.

5. Forward entropy and indecomposability

In this section we will see how, under certain hypotheses, positive forward entropy implies an indecomposable subcontinuum in the inverse limit. First, we need to put a tree structure on \( W \). Forward entropy and indecomposability are two sections.

Theorem 12. [4] If \( \mathcal{W}^F ( (f_i)_{i=0}^{\infty}, \mathcal{A}) \) is invariant under shifts and \( \text{Ent}(\mathcal{W}^F ( (f_i)_{i=0}^{\infty}, \mathcal{A})) > 0 \), then there exists a collection of integers \( G \), and intervals \( A, B \in \mathcal{A} \) such that \( \mathcal{W}^F ( (f_i)_{i=0}^{\infty}, \mathcal{A}, G) \) is complete on \( \{A, B\} \).

Theorem 13. (Ye [5]) Suppose \( \{f_i\}_{i=0}^{\infty} \) is a collection of onto maps of the interval. If there exist subintervals \( \{A_i\}_{i=0}^{\infty}, \{B_i\}_{i=0}^{\infty} \) such that \( \text{int}(A_i \cap B_i) = \emptyset \) and an increasing collection of positive integers \( \{n_i\}_{i=0}^{\infty} \) such that \( A_i \cup B_i \subset f_{n_i+1}^{g(i)} (A_{i+1}) \cap f_{n_i+1}^{g(i)} (B_{i+1}) \) for all \( i \), then the inverse limit \( X = \lim \{I, f_i\}_{i=1}^{\infty} \) must contain an indecomposable subcontinuum.

The intervals \( A, B \in \mathcal{A} \) and sequence \( G = (g(i))_{i=1}^{\infty} \) that satisfy Theorem 12 will now be inspected. Let \( \mathcal{D} \) be the set of subarcs \( D_\alpha \) with the following properties:

1. \( \text{int}(A \cap f_{g(1)}^{g(2)} (A \cap D_\alpha)) \neq \emptyset \);
2. \( \text{int}(A \cap f_{g(1)}^{g(2)} (B \cap D_\alpha)) \neq \emptyset \);
3. \( \text{int}(B \cap f_{g(1)}^{g(2)} (A \cap D_\alpha)) \neq \emptyset \);
4. \( \text{int}(B \cap f_{g(1)}^{g(2)} (B \cap D_\alpha)) \neq \emptyset \).

Partially order \( \mathcal{D} \) by inclusion and let \( \mathcal{P} \) be some maximum chain in the ordering. Let \( D = \bigcap_{D_\alpha \in \mathcal{P}} D_\alpha. \)

Theorem 14. Suppose that \( \{f_i\}_{i=1}^{\infty} \) is a collection of maps on \( I \), and \( A \) and \( B \) are sub-intervals and \( G \) is an increasing sequence of positive integers such that \( \mathcal{W}^F ( (f_i)_{i=0}^{\infty}, \mathcal{A}, G) \) is complete on \( \{A, B\} \). If \( D \) (as defined previously) is a nondegenerate arc, then \( X = \lim \{I, f_i\}_{i=1}^{\infty} \) must contain an indecomposable subcontinuum.

Proof. Let \( C_1 \) be the arc irreducible about \( A \cup B \). Then there must exist a minimal subarc \( C_2 \) of \( C_1 \) such that

\[ (A_2) \quad D \subset f_{g(2)}^{g(3)} (A \cap C_2); \]
\[ (B_2) \quad D \subset f_{g(2)}^{g(3)} (B \cap C_2). \]

If \( (A_2) \) is false, then one of the following must be true
(1) $\{A, A, A\} \notin \Pi_3(\mathcal{W}^F(f_i)_{i=0}^\infty, \{A, B\}, G)$;
(2) $\{A, B, A\} \notin \Pi_3(\mathcal{W}^F(f_i)_{i=0}^\infty, \{A, B\}, G)$;
(3) $\{B, A, A\} \notin \Pi_3(\mathcal{W}^F(f_i)_{i=0}^\infty, \{A, B\}, G)$;
(4) $\{B, B, A\} \notin \Pi_3(\mathcal{W}^F(f_i)_{i=0}^\infty, \{A, B\}, G)$;

which is impossible since $\mathcal{W}^F(f_i)_{i=0}^\infty, A, G)$ is complete on $\{A, B\}$. Thus, statement $(A_2)$ must be true. In a similar manner, statement $(B_2)$ is also true. Continuing inductively, suppose that $C_2, \ldots, C_{k-1}$ have been found. Then there exists a minimal subarc $C_k$ of $C_{k-1}$ such that

$$(A_k) \quad C_{k-1} \subset f_{g(k-1)}^g(A \cap C_k);$$

$$(B_k) \quad C_{k-1} \subset f_{g(k-1)}^g(B \cap C_k).$$

Statements $(A_k)$ and $(B_k)$ are true for the same reasons as statements $(A_2)$ and $(B_2)$. Also, since $D$ is nondegenerate by hypothesis, each $C_k$ is nondegenerate by hypothesis. Therefore, by Theorem 13 (where $A_i = A \cap C_i$ and $B_i = B \cap C_i$), $X$ must have an indecomposable subcontinuum. □

The following corollaries are the main results of this section.

**Corollary 15.** If $\overline{A} \cap \overline{B} = \emptyset$ then $X$ contains a nondegenerate indecomposable subcontinuum.

**Proof.** Follows from the fact that $D$ must intersect both $\overline{A}$ and $\overline{B}$ and thus is nondegenerate. □

**Corollary 16.** If $f_{g(1)}^{(2)}$ has a finite number of monotone pieces on $A \cup B$ and $\overline{A} \cap \overline{B}$ is totally disconnected, then $X$ contains a nondegenerate indecomposable subcontinuum.

**Proof.** If $D \not\subset \overline{A} \cap \overline{B}$ then $D$ must contain at least 2 points since $D \cap \overline{A} \neq \emptyset$ and $D \cap \overline{B} \neq \emptyset$. Hence, $D$ is a nondegenerate arc. For purposes of a contradiction, suppose that $D = \{a\} = \overline{A} \cap \overline{B}$. Then there exists an $\epsilon > 0$ such that $(a - \epsilon, a) \subset A$ and $(a, a + \epsilon) \subset B$ (or vice versa). Since $f_{g(1)}^{(2)}$ has only a finite number of monotone pieces, $\epsilon$ can be chosen small enough such that at least 1 of the following is true:

1. $f_{g(1)}^{(2)}((a - \epsilon, a))) \cap A = \emptyset$;
2. $f_{g(1)}^{(2)}((a - \epsilon, a))) \cap B = \emptyset$;
3. $f_{g(1)}^{(2)}((a, a + \epsilon)) \cap A = \emptyset$;
4. $f_{g(1)}^{(2)}((a, a + \epsilon)) \cap B = \emptyset$.

Since $D$ is a singleton, there exists a $D_a \subset (a - \epsilon, a + \epsilon)$. However, that $D_a$ will violate one of the 4 properties of $D$ which is a contradiction. Therefore, $D$ must be a nondegenerate arc and we can apply Corollary 15. □

**Corollary 17.** If $f_{g(1)}^{(i+1)}$ has a finite number of monotone pieces on $A \cup B$ for some $i$ and $A \cap B = \emptyset$, then $X$ contains a nondegenerate indecomposable subcontinuum.

Same as Corollary 16. Just have $D$ defined for $f_{g(1)}^{(i+1)}$.

**Corollary 18.** Suppose $\{f_i\}_{i=0}^\infty$ is a collection of maps on $I$ such that each $f_i$ has a finite number of monotone pieces and let $A$ be a finite closed cover on $I$. If $\mathcal{W}^F(\{f_i\}_{i=0}^\infty, A)$ is closed under shifts and $\text{Ent}_F(\{f_i\}_{i=0}^\infty, A) > 0$, then $X = \lim_{i \to \infty}(I, f_i)_{i=0}^\infty$ must contain an indecomposable subcontinuum.

The following examples show that Theorem 14 is the sharpest result on the relationship between positive forward entropy and indecomposability.
Example 19. Define

\[
h_0(x) = \begin{cases} 
\frac{1}{2} + \frac{1}{\sqrt{3}} (2x - 1)^{4/3} \sin \left( \frac{\pi}{3(x-1/2)} \right) & x \neq 1/2, \\
1/2 & x = 1/2,
\end{cases}
\]

and

\[
h_1(x) = \begin{cases} 
\frac{1}{2} h_{i-1}(3x) & x \in [0, \frac{1}{3}), \\
\frac{1}{4} + \frac{1}{2} h_{i-1}(3(x - \frac{1}{3})) & x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
\frac{1}{4} + \frac{1}{2} h_{i-1}(3(x - \frac{2}{3})) & x \in (\frac{2}{3}, 1].
\end{cases}
\]

(See Fig. 4.)

Interval \([a, b]\) is a perturbation component of \(h_i\) if for every \(y \in h_i([a, b]), h_i^{-1}(y)\) is disconnected. \([a, b]\) is a maximal perturbation component if every interval that contains \([a, b]\) properly is not a perturbation component. Notice that \(h_i\) has 3\(^3\) maximal perturbation components. Since the diameter of each perturbation component shrinks in \(h_i\) as \(i \to \infty\) and the image of each perturbation component contracts when it is mapped under \(h_i\), with some work the inverse limit of \([h_i]_{i=0}^{\infty}\) can be shown to be an arc. However, by letting \(A = \{[0, 1/2], [1/2, 1]\}\), it is clear that \(\mathcal{W}^F([h_i]_{i=0}^{\infty}, A) = \{\langle A_i \rangle_{i=0}^{\infty} | A_i \in A\}\), which is closed under shifts and has positive entropy.

Example 20. For \(0 < \epsilon < 1/6\), let \(\hat{\tau}_\epsilon : [1/3 - 2\epsilon, 2/3 + 2\epsilon] \to [1/3 - 2\epsilon, 2/3 + 2\epsilon]\) be a piecewise linear map with endpoints (in order) \((1/3 - 2\epsilon, 1/3 - 2\epsilon), (1/3 - \epsilon, 2/3 + 2\epsilon), (1/3, 2/3), (2/3, 1/3), (2/3 + \epsilon, 1/3 - 2\epsilon), (2/3 + 2\epsilon, 2/3 + 2\epsilon)\). Let \(\epsilon_k = 1/2^{k+1}\) and for \(2^k \leq i < 2^{k+1}\) define

\[
r_i(x) = \begin{cases} 
x & x \in [0, 1/3 - 2\epsilon_k] \cup (2/3 + 2\epsilon_k, 1], \\
\hat{\tau}_{\epsilon_k}(x) & x \in [1/3 - 2\epsilon_k, 2/3 + 2\epsilon_k].
\end{cases}
\]

(See Fig. 5.)

Let \(\mathcal{A}_\epsilon = \{[1/3 - 2\epsilon, 1/3], [2/3, 2/3 + 2\epsilon]\}\). Then since

\[
\hat{\tau}_\epsilon([1/3 - 2\epsilon, 1/3]) = \hat{\tau}_\epsilon([2/3, 2/3 + 2\epsilon]) = [1/3 - 2\epsilon, 2/3 + 2\epsilon],
\]

\(|\Pi_k(\mathcal{W}^F(\hat{\tau}_\epsilon, \mathcal{A}_\epsilon))| = 2^k\). Let \(A = \{[0, 1/3], [2/3, 1]\}\), then notice that \([1/3 - 2\epsilon, 1/3] \subset [0, 1/3]\) and \([2/3, 2/3 + 2\epsilon] \subset [2/3, 1]\). So it follows that

\[
|\Pi_{2^k}(\mathcal{W}^F([r_i]_{i=2^k}^{2^{k+1}}, \mathcal{A}))| = |\Pi_{2^k}(\mathcal{W}^F([r_i]_{i=2^k}^{2^{k+1}}, \mathcal{A}_\epsilon))| = 2^k.
\]

Hence

\[
|\Pi_{2^{k+1}}(\mathcal{W}^F([r_i]_{i=2^k}^{2^{k+1}}, \mathcal{A}))| \geq 2^k.
\]

So \(\text{Ent}([r_i]_{i=0}^{\infty}, \mathcal{A}) \geq (1/2) \log(2)\) and from Theorem 8 it follows that \(\text{Ent}([r_i]_{i=0}^{\infty}) > 0\). However, by construction, not only is \(r_i([1/3 - 2\epsilon_{k+1}, 2/3 + 2\epsilon_{k+1}]) \subset [1/3 - 2\epsilon_k, 2/3 + 2\epsilon_k]\) but also \(r_{2^{k+1}}([1/3 - 2\epsilon_{k+1}, 2/3 + 2\epsilon_{k+1}]) \subset [1/3 - 2\epsilon_k, 2/3 + 2\epsilon_k]\). So there is no infinitely continuing recursive crookedness as \(\epsilon_k \to 0\) and the inverse limit does not contain an indecomposable subcontinuum. In fact, it can be shown that \(X = \lim [I, r_i]_{i=0}^{\infty}\) is 2 rays limiting to an arc. Also, one can verify that \(\mathcal{W}^F([r_i]_{i=0}^{\infty}, \mathcal{A})\) is not invariant under shifts.
6. Backwards entropy and local connectivity

In this section, we look at the relationship between backwards entropy and local connectivity. $X$ is locally connected if for every $x \in X$ and for every open set $U$ that contains $x$ there exists a connected open set $V$ such that $x \in V \subset U$. The first proposition gives a condition when a subset of an inverse limit is not connected.

**Proposition 21.** Suppose that

$$X = \lim_{\leftarrow} \{ I, f_i \}_{i=0}^{\infty}.$$  

If $V \subset X$ and $\pi_i(V)$ is disconnected for some $i$, then $V$ is disconnected.

The next lemma shows when an inverse limit is not locally connected.

**Lemma 22.** Let $f_i : I \rightarrow I$ and $C = [a, b]$ where $0 < a < b < 1$. Suppose $C_i$ is the maximum collection of subarcs with the following properties:

1. If $A \in C_i$ then $f_i(A) = C$.
2. If $B$ is a proper subset of $A \in C_i$, then $f_i(B) \neq C$.

Then if $|C_i| \rightarrow \infty$ as $i \rightarrow \infty$,

$$X = \lim_{\leftarrow} \{ I, f_i \}_{i=0}^{\infty}$$

is not locally connected.

**Proof.** Since $I$ is compact, $f_i$ must be uniformly continuous. Hence $|C_i|$ must be finite. For every $A \in C_i$, there is a minimal subarc $B$ of $(f_i)^{-1}(A)$ such that $f_i(A) = B$. It follows that $B \in C_{i+1}$. Conversely, for every $B' \in C_{i+1}$ there exists a unique $A' \in C_i$ such that $f_i(A') = B$. Since $|C_i| \rightarrow \infty$ there must exist a subsequence $(i_j)_{j=1}^{\infty}$ and a collection $(A_{i_j})_{j=1}^{\infty}$ such that

1. $A_{i_j} \in C_{i_j}$,
2. $f_{i_j}^{-1}(A_{i_{j+1}}) = A_{i_j}$,
3. There exists a $B_{i_{j+1}} \in C_{i_{j+1}}$, where int($B_{i_{j+1}} \cap A_{i_{j+1}}$) = $\emptyset$, such that $f_{i_j}^{-1}(B_{i_{j+1}}) = A_{i_j}$. (Otherwise, sup $|C_i|$ would be finite.)
Notice that by (ii) if $C_{ij} \in C_{ij}$ and $C_{i,j+1} \in C_{i,j+1}$ such that $f_{ij}^{j+1}(C_{i,j+1}) = C_{ij}$ then $f_{ij}^{j+1}$ maps endpoints of $C_{i,j+1}$ to endpoints of $C_{ij}$. Let $a < c < b$ and $\{c_{ij}\}_{j=1}^{\infty}$ be a collection such that $c_{ij} \in A_{ij}$, $f_{ij}^{1}(c_{ij}) = c$ and $f_{ij}^{j+1}(c_{ij}) = c_{ij}$. Then define $c$ to be the point such that $\pi_{ij}(c) = c_{ij}$ for each $j \geq 1$. Choose $\epsilon > 0$ to be such that $a < c - \epsilon < c < c + \epsilon < b$ and let $U = \pi_{0}^{1}((c - \epsilon, c + \epsilon))$.

Claim 23. If $V$ is an open set such that $c \in V \subset U$ then $V$ is disconnected.

For each $j$, if $U'$ is a component of $\pi_{ij}(U)$ that intersects any $C_{ij} \in C_{ij}$, then $U'$ must be contained in $C_{ij}$. Otherwise, an endpoint of $C_{ij}$ must be contained in $U'$ which would imply that either $a$ or $b$ is in $\pi_{0}(U)$ which is impossible. Next, there exists a positive integer $J$ and $\delta > 0$ such that $\pi_{ij}^{-1}((c_{ij} - \delta, c_{ij} + \delta)) \subset V$. Consider $A_{ij}, B_{ij}$ found in (i)–(iii). Since $c \in \pi_{ij}^{-1}((c_{ij} - \delta, c_{ij} + \delta))$, it follows that $\pi_{ij+1}(c) = c_{ij+1} \in A_{ij+1}$. So $A_{ij+1} \cap \pi_{ij+1}^{-1}((c_{ij} - \delta, c_{ij} + \delta)) \neq \emptyset$. Also, there exists $b_{ij+1} \in B_{ij+1}$ such that $f_{ij}^{j+1}(b_{ij+1}) = c_{ij}$. Hence $b_{ij+1} \in \pi_{ij+1}^{-1}((c_{ij} - \delta, c_{ij} + \delta))$. Therefore $B_{ij+1} \cap \pi_{ij+1}^{-1}((c_{ij} - \delta, c_{ij} + \delta)) \neq \emptyset$. It then follows that since the components of $\pi_{ij}(V)$ are subsets of the components of $\pi_{ij}(U)$, there exist a component of $\pi_{ij+1}(V)$ contained in $A_{ij+1}$ and a component of $\pi_{ij+1}(V)$ contained in $B_{ij+1}$. Hence, $\pi_{ij+1}(V)$ is disconnected and thus $V$ is disconnected.

Therefore, $X$ is not locally connected at $c$ and Lemma 22 follows. □

Lemma 24. Let $A$ and $B$ be open covers of $I$. If $\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, A) = 0$ and $\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, B) = 0$, then $\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, A \cup B) = 0$.

Proof. Since $(f_{i}^{j+1})^{-1}(A_{i}) \cap (f_{i}^{j+1})^{-1}(B_{i}) = (f_{i}^{j})^{-1}(A_{i} \cap B_{i})$, it follows that

$$\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A \cup B) = \bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A) \vee \bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(B).$$

Let $A'$ and $B'$ be subcovers of $\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A)$ and $\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(B)$, respectively, such that $|A'| = N(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A))$ and $|B'| = N(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(B))$. $A' \cup B'$ is a subcover of $\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A \cup B)$, so

$$N\left(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A)\right)N\left(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(B)\right) = |A'| \leq |A' \cup B'| \geq |B'| \geq N\left(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A \cup B)\right).$$

Therefore,

$$\limsup_{n \to \infty} \frac{\log N(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A))}{n} + \limsup_{n \to \infty} \frac{\log N(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(B))}{n} = 0$$

implies

$$\limsup_{n \to \infty} \frac{\log N(\bigvee_{i=0}^{n}(f_{i}^{j})^{-1}(A \cup B))}{n} = 0,$$

and the lemma follows. □

Theorem 25. If $\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, I) > 0$, then there exist $a < b$ such that

$$\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, \{0, b, (a, 1]\}) > 0.$$

Proof. There exists a collection of open intervals $U = \{U_{k}\}_{k=1}^{n}$ such that

1. $\text{Ent}_{B}(\{f_{i}\}_{i=0}^{\infty}, U) > 0$;
2. $\inf(U_{k}) < \inf(U_{k+1}) < \sup(U_{k}) < \inf(U_{k+2}) < \sup(U_{k+1}) < \sup(U_{k+2})$ for each $k$;
3. $\inf(U_{1}) = 0$;
(4) $\sup(U_n) = 1$.

If $|U| = 2$, we are done. So consider the next claim.

**Claim 26.** If $\text{Ent}_B((f_i)_{i=0}^{\infty}, \mathcal{U}) > 0$ and $|U| > 2$ then there is a collection of open intervals $\mathcal{V}$ such that $\mathcal{U}$ refines $\mathcal{V}$, $|\mathcal{U}| > |\mathcal{V}|$ and $\text{Ent}_B((f_i)_{i=0}^{\infty}, \mathcal{V}) > 0$.

Let $A = \{U_1 \cup U_2, U_3 \cup U_4, \ldots, U_{n-1} \cup U_n\}$ if $n$ is even and $A = \{U_1 \cup U_2, \ldots, U_{n-2} \cup U_{n-1}, U_n\}$ if $n$ is odd. Also, let $B = \{U_1, U_2 \cup U_3, \ldots, U_{n-2} \cup U_{n-1}, U_n\}$ if $n$ is even and $B = \{U_1, U_2 \cup U_3, \ldots, U_{n-1} \cup U_n\}$ if $n$ is odd. Then $A \cup B = \mathcal{U}$. It follows from Lemma 24 that $\text{Ent}_B((f_i)_{i=0}^{\infty}, A) > 0$ or $\text{Ent}_B((f_i)_{i=0}^{\infty}, B) > 0$. Let $\mathcal{V} = B$ if $\text{Ent}_B((f_i)_{i=0}^{\infty}, A) > 0$ and $\mathcal{V} = B$ otherwise. Then $\mathcal{V}$ satisfies the conclusion of the claim.

It follows from repeated use of the claim that there exists a collection $\mathcal{W}$ of 2 open intervals such that $\text{Ent}_B((f_i)_{i=0}^{\infty}, \mathcal{W}) > 0$. $\square$

Let $C$ be a finite cover of $X$. $A \subset X$ is a sufficient cover set for $(C, X)$ provided that for every $S \subset C$, if $S$ covers $A$ then $S$ covers $X$.

**Lemma 27.** For every finite cover $C$ of $X$, there exists a finite sufficient cover set.

**Proof.** Let $\mathcal{U} = \{S; S \subset C \text{ and } S \text{ does not cover } X\}$. Since $C$ is finite, $\mathcal{U}$ must be finite. For each $S \in \mathcal{U}$, choose $x_S \in X - \bigcup_{S \in \mathcal{S}} S$. Let $A = \{x_S; S \in \mathcal{U}\}$. Then $|A| \leq |\mathcal{U}|$, so $A$ is finite. Furthermore, if $\mathcal{V}$ is a subset of $C$ that covers $A$, then $\mathcal{V} \notin \mathcal{U}$ and hence $\mathcal{V}$ must cover $X$. Thus, $A$ is a sufficient cover set. $\square$

A continuous function $f : X \to Y$ is weakly-confluent provided for every subcontinuum $K$ of $f(X)$, there exists a component $C$ of $f^{-1}(K)$ such that $f(C) = K$. The following well-known fact will be used in the proof of the next theorem: Every continuous map of a continuum $X$ onto $I$ is weakly-confluent.

**Theorem 28.** Suppose $a < b$ and let $A = ([0, b), (a, 1])$. If $\text{Ent}_B((f_i)_{i=0}^{\infty}, A) > 0$, then $X$ is not locally connected.

**Proof.** Let $(x(i))_{i=1}^{m_n}$ be a finite sufficient cover set for $V_n = (f_0^{-1}(f_0^{i0})^{-1}(A), I)$ ordered such that $x(i) < x(i+1)$.

Let $i_0 = 1$ and $i_1$ be the smallest integer in $i_0 + 1, \ldots, m_n$ such that there exists $j_1 \in \{0, \ldots, n-1\}$ such that $f_0^{i_1h}(x(i_1)) \in [0, a]$ and $f_0^{i_1h}(x(i_0)) \in [b, 1]$ (or vice versa).

Then $i_0, i_1, \ldots, i_{k-1}$ being found, let $i_k$ be the smallest integer in $\{i_{k-1} + 1, \ldots, m_n\}$ such that there exist $j_k \in \{0, \ldots, n-1\}$ such that $f_0^{i_kh}(x(i_{k-1})) \in [0, a]$ and $f_0^{i_kh}(x(i_k)) \in [b, 1]$ (or vice versa).

Next, let $E_k = (x(i_k), x(i_{k+1}), \ldots, x(i_{k+1} - 1))$ and define $p_n$ to be the integer such that $x(m_n) \in E_{p_n}$. By construction, for each $j$ one of the following must be true:

1. $f_0^{i_jh}(x(i_j)) \in [0, b)$ for every $x(i_j) \in E_k$. If this is true let $Y_j = [0, b)$.
2. $f_0^{i_jh}(x(i_j)) \in (a, 1]$ for every $x(i_j) \in E_k$. If this is true and (1) is false let $Y_j = (a, 1]$.

Then $E_k \subset \bigcap_{j=0}^{n-1} (f_0^{i_jh})^{-1}(Y_j) \in V_n = (f_0^{-1}(f_0^{i_0h})^{-1}(A), I)$. Let $\mathcal{Y} = \{\bigcap_{j=0}^{n-1} (f_0^{i_jh})^{-1}(Y_j)\}_{k=0}^{p_n}$. Hence $p_n = |\mathcal{Y}| + 1$. However, $\mathcal{Y}$ is a cover of sufficient cover set $(x(i))_{i=1}^{m_n} = \bigcup_{k=0}^{p_n} E_k$. Therefore, $\mathcal{Y}$ must cover $I$. Thus, $\mathcal{Y}$ satisfies the conclusion of the claim.

Since $\mathcal{Y}$ is a cover of sufficient cover set $(x(i))_{i=1}^{m_n} = \bigcup_{k=0}^{p_n} E_k$. Therefore, $\mathcal{Y}$ must cover $I$. Thus, $\mathcal{Y}$ satisfies the conclusion of the claim.

Then from Theorems 25 and 28, we get the following corollary which is the main result of this section:
Corollary 29. If $\text{Ent}_B(\{f_i\}_{i=0}^\infty, I) > 0$, then

$$X = \lim_{\leftarrow} \{I, f_i\}_{i=1}^\infty$$

is not locally connected.

A continuum is Suslinean if there does not exist an uncountable number of disjoint non-degenerate subcontinua. The next example shows that Corollary 29 is sharp.

Example 30. Let $w_n : I \rightarrow I$ be defined by

$$w_n(x) = \begin{cases} 3x & x \in [0, \frac{1}{3}), \\ 1 - 3(x - (\frac{1}{3} + \frac{k}{3^n+1})) & x \in [\frac{1}{3} + \frac{k}{3^n+1}, \frac{1}{3} + \frac{k+1}{3^n+1}] \text{ and } k \text{ even}, \\ 1 - \frac{1}{3^n} + 3(x - (\frac{1}{3} + \frac{k}{3^n+1})) & x \in [\frac{1}{3} + \frac{k}{3^n+1}, \frac{1}{3} + \frac{k+1}{3^n+1}] \text{ and } k \text{ odd}, \end{cases}$$

where $0 \leq k \leq 2(3^n) - 1$. (See Fig. 6.)

If $\mathcal{A} = \{[0, 1/3], [1/3, 2/3], [2/3, 1]\}$ then one can see that $W_B^B(\{w_i\}_{i=0}^\infty, \mathcal{A})$ is closed under shifts. Furthermore, $\text{Ent}_B(\{w_i\}_{i=0}^\infty, \{[0, 1/3], [2/3, 1]\}) > 0$ which means that $\text{Ent}_B(\{w_i\}_{i=0}^\infty) > 0$. However,

$$X = \lim_{\leftarrow} \{I, w_i\}_{i=0}^\infty$$

is a ray limiting to an arc which is Suslinean.

On the other hand, in Example 19, it can be shown that

$$\text{Ent}_B(\{h_i\}_{i=0}^\infty, \{[0, 1/2], [1/2, 1]\}) > 0.$$

However, the inverse limit is just an arc. Of course, by Corollary 29 the entropy on open sets must still be 0.

References