Abstract. It is shown that if $X$ is a chainable continuum and $h : X \to X$ is a homeomorphism such that the topological entropy of $h$ is greater than 0, then $X$ must contain an indecomposable subcontinuum. This answers a question of Barge.

1. Introduction

In dynamics, topological entropy is a number in $[0, \infty]$ that gives a measure of the chaotic behavior of a function on a space. The connection between entropy and the dynamics of a continuous function is well documented. Many simple functions such as the tent map have positive entropy. However, for a homeomorphism to have positive entropy, it appears that the local structure must be complex. For example, in [5] it is shown that no homeomorphism of a regular continuum can have positive entropy. In [7] and [8], Ye showed that homeomorphisms of hereditarily decomposable chainable continua that are induced by square commuting diagrams on inverse systems of intervals must have zero entropy. In this paper, it is shown that every chainable continuum that admits a positive entropy homeomorphism must contain a nondegenerate indecomposable subcontinuum. Indecomposable continua are known to have a very complicated local structure. This answers a question due to Barge [2].

A continuum $X$ is a compact connected metric space. A map is a continuous function. A continuum $X$ is decomposable provided there exist proper subcontinua $H$ and $K$ such that $X = H \cup K$. A continuum is indecomposable if it is not decomposable. If $\mathcal{U}$ is a collection of open sets, the mesh of $\mathcal{U}$ is defined as $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) : U \in \mathcal{U}\}$. A chain $C$ is an indexed collection of open sets $\{C_1, C_2, ..., C_n\}$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Here, $C_1$ and $C_n$ are the endlinks of the chain. A continuum $X$ is chainable if for every $\epsilon > 0$ there exists a chain covering $X$ such that $\text{mesh}(C) < \epsilon$. Chainable continua are also called arc-like and snake-like continua.

Let $X$ be a compact metric space, $f : X \to X$ be a map, and $\mathcal{U}$ be a finite open cover of $X$. Define $N(\mathcal{U})$ be the number of sets in a finite subcover of $\mathcal{U}$ with smallest cardinality. If $\mathcal{U}$ and $\mathcal{V}$ are two open covers of $X$, let $\mathcal{U} \lor \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$. Also, define

$$\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U}) = \mathcal{U} \lor f^{-1}(\mathcal{U}) \lor ... \lor f^{-n+1}(\mathcal{U}),$$

where $f^0 = \text{id}$ and

$$\text{Ent}(f, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U})).$$

Then the topological entropy of $f$ is defined as

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$\text{Ent}(f) = \sup\{\text{Ent}(f, \mathcal{U}, X) : \mathcal{U} \text{ is an open cover of } X\}.$

If $\mathcal{U}$ and $\mathcal{V}$ are finite open covers of $X$ such that for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$, then $\mathcal{V}$ refines $\mathcal{U}$. If for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $\overline{V} \subset U$, then $\mathcal{V}$ closure refines $\mathcal{U}$.

The following propositions are well known and can be found in several texts such as [3] and [6].

**Proposition 1.** If $\mathcal{V}, \mathcal{U}$ are open covers of $X$ such that $\mathcal{V}$ refines $\mathcal{U}$, then $\text{Ent}(f, \mathcal{V}) \geq \text{Ent}(f, \mathcal{U})$.

**Proposition 2.** For each positive integer $k$, $\text{Ent}(f^k) = k\text{Ent}(f)$.

To prove the main result, the following steps are taken (terms in italics will be defined later):

1. Entropy is defined on collections of infinite words on finite alphabets.
2. Relative entropy between 2 symbols in the alphabet is defined.
3. If a collection of infinite words has positive entropy, then it is shown through combinatorical techniques that “embedded” in the collection of words is a subcollection of words on 2 symbols that is complete.
4. Collections of words are then generated by finite open covers and inverse images of continuous function $f$.
5. Combinatorical techniques are used to show that if $\text{Ent}(f) > \log(2)$, then there exist disjoint open sets $A, B$ whose relative entropy is positive.
6. It is shown that for every $\epsilon > 0$ there exist a chain that is crooked between $A$ and $B$ whose endlinks have positive relative entropy.
7. Using the well-used techniques of Barrett [1], it is shown that there must be an indecomposable subcontinuum.

### 2. The Entropy of Words

For a more complete treatment on the entropy of words see [4]. An alphabet $A$ is a finite set of symbols. A word is a sequence composed from the elements of $A$. Words can be finite, $W_n = \langle A_i \rangle_{i=1}^n$, or infinite, $W = \langle A_i \rangle_{i=1}^\infty$. The length of a word is the number of elements in the sequence. Define $\pi_k(\langle A_i \rangle_{i=1}^\infty) = A_k$ and $\Pi_k(\langle A_i \rangle_{i=1}^\infty) = \langle A_i \rangle_{i=1}^k$. Likewise, define $\pi_k(\langle A_i \rangle_{i=1}^n) = A_k$ and $\Pi_k(\langle A_i \rangle_{i=1}^n) = \langle A_i \rangle_{i=1}^k$ if the word is finite and $k \leq n$. If $W$ is a word, then $\Pi_k(W)$ is called the prefix of length $k$ of $W$. If $W_n$ is a word of length $n$ then $\langle W_n \rangle$ is a new word of length $n + 1$ formed by adding the symbol $A$ to the end of $W_n$.

$\mathcal{W}(A)$ ($\mathcal{W}$ when there is no confusion) will represent some collection of infinite words on $A$. If $B \subset A$, then

$$\mathcal{W}(B) = \{\langle B_i \rangle_{i=1}^\infty | B_i \in B\}.$$  

Likewise, $\mathcal{W}_n$ will represent some collection of words of length $n$. Next, define $\Pi_k(\mathcal{W}) = \{\Pi_k(W) | W \in \mathcal{W}\}$. $\Pi_k(\mathcal{W}_n)$ can be defined in a similar manner provided $n \leq k$.

It will be useful to define distance between words. Let $A, B \in A$ and let

$$d_A(A, B) = \begin{cases} 
0 & A = B \\
1 & A \neq B
\end{cases}$$
Then define the distance between \( \langle A_i \rangle_{i=1}^\infty, \langle B_i \rangle_{i=1}^\infty \in \mathcal{W} \) by

\[
d(\langle A_i \rangle_{i=1}^\infty, \langle B_i \rangle_{i=1}^\infty) = \sum_{i=1}^\infty \frac{d(A_i, B_i)}{2^i}.
\]

Now we can define \( \mathcal{W} \) to be closed in the topological sense. That is, \( \mathcal{W} \) is closed provided it contains all of its limit points. Topologically, \( \mathcal{W} \) is a totally disconnected metric space. This topological structure will allow us to prove some theorems more easily, but is not directly important in the definition of entropy and the application of words to functions in later sections. The most useful fact that follows from this definition of closed is the following proposition.

**Proposition 3.** Suppose that \( \mathcal{W} \) is a closed collection of infinite words on \( \mathcal{A} \) and \( \{g(i)\}_{i=1}^\infty \) is an increasing sequence of positive integers. If \( \Pi_{g(i)}(W) \in \Pi_{g(i)}(\mathcal{W}) \) for all \( i \), then \( W \in \mathcal{W} \).

**Proof.** The proposition follows from the fact that for each \( i \), there exists \( W^i \in \mathcal{W} \) such that \( \Pi_{g(i)}(W^i) = \Pi_{g(i)}(W) \). Hence \( W \) is a limit point of \( \{W^i\}_{i=1}^\infty \) and therefore \( W \in \mathcal{W} \). \( \square \)

Let \( \mathcal{B} \subset \mathcal{A} \). A collection of words \( \mathcal{W} \) is **complete** on subalphabet \( \mathcal{B} \) provided that every possible word composed from \( \mathcal{B} \) is in \( \mathcal{W} \). That is, if \( B_i \in \mathcal{B} \), then \( \langle B_i \rangle_{i=1}^\infty \in \mathcal{W} \). Likewise, \( W_k \) is complete on \( \mathcal{B} \) provided if \( B_i \in \mathcal{B} \), then \( \langle B_i \rangle_{i=1}^k \in \mathcal{W} \). Clearly, if \( \mathcal{W} \) is complete on \( \mathcal{B} \), then \( \mathcal{W}(\mathcal{B}) \) is complete on \( \mathcal{B} \). Also notice that if \( \mathcal{W}(\mathcal{B}) \) is complete on \( \mathcal{B} \), then \( \mathcal{W}(\mathcal{B}) \) takes on the structure of a \(|\mathcal{B}|\)-ary tree.

\( \mathcal{W} \) is **shift-invariant** provided \( \langle A_i \rangle_{i=1}^\infty \in \mathcal{W} \) implies \( \sigma^k(\langle A_i \rangle_{i=1}^\infty) = \langle A_i \rangle_{i=k+1}^\infty \in \mathcal{W} \) for each \( k \geq 0 \). Here, \( \sigma \) is called a shift of \( \langle A_i \rangle_{i=1}^\infty \).

Suppose that \( G = \{g(i)\}_{i=1}^\infty \) is a strictly increasing sequence of positive integers \( G_n = \{g(i)\}_{i=1}^n \) for finite sequences). Then define

\[
(W)_G = \{ (\pi_{g(i)}(W))_{i=1}^\infty \mid W \in \mathcal{W} \}.
\]

Define \( (W)_{G_n} \) and \( (W_k)_{G_n} \) in a similar manner. Again, if \( \mathcal{B} \subset \mathcal{A} \) then

\[
(W(\mathcal{B}))_G = \{ (B_i)_{i=1}^\infty \in (W)_{G} \mid B_i \in \mathcal{B} \}.
\]

The following propositions now follow quickly.

**Proposition 4.** If \( (W)_{G} \) is complete on \( \mathcal{B} \) and \( G' \) is a subsequence of \( G \), then \( (W)_{G'} \) is complete on \( \mathcal{B} \).

**Proposition 5.** If \( (W)_{G} \) is complete on \( \mathcal{A} \) and \( \mathcal{B} \subset \mathcal{A} \), then \( (W)_{G} \) is complete on \( \mathcal{B} \).

**Proposition 6.** If \( \mathcal{W} \) is a closed collection of infinite words, then \( (W)_{G} \) is a closed collection of infinite words.

**Proof.** Given increasing sequence \( G = \{g(i)\}_{i=1}^\infty \) and \( W \in \mathcal{W} \), define \( F_G(W) = (\pi_{g(i)}(W))_{i=1}^\infty \). Then clearly, \( F_G \) is a continuous function from \( \mathcal{W} \) onto \( (W)_{G} \). Since \( \mathcal{W} \) is closed, then its continuous image \( (W)_{G} \) must also be closed. \( \square \)

Finally, **entropy** of an infinite collection of words \( \mathcal{W} \) is defined as

\[
\text{ENT}(\mathcal{W}) = \lim_{k \to \infty} \sup \frac{\log |\Pi_k(\mathcal{W})|}{k}.
\]
Notice that if $W$ is a complete collection of words on $B \subset A$, then $|\Pi_k(W)| \geq |B|^k$ and it follows that
$$\log |B| \leq \text{ENT}(W) \leq \log |A|.$$ 
Suppose that $W$ is a collection of infinite words on $A$ and that $B \subset A$. Again, let $G = \{g(i)\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. Define the entropy of $W$ relative to $(B, G)$ by
$$\text{ENT}(W, B, G) = \limsup_{i \to \infty} \frac{\log |\Pi_i((W(B))_G)|}{g(i)},$$
and the entropy of $W$ relative to $B$ by
$$\text{ENT}(W, B) = \sup_G \{\text{ENT}(W, B, G)\}.$$

Notice that $(W)_G$ itself is a complete collection of words and it will be useful to consider $\text{ENT}((W)_G)$. Also, since $|\Pi_{g(i)}((W)_G)| \geq |\Pi_i((W)_G)|$, then $\text{ENT}((W)_G) \geq \text{ENT}(W, G)$.

The following 2 theorems are Theorem 14 and Corollary 20 respectively in [4].

**Theorem 7.** Suppose that $W$ is a closed shift-invariant collection of infinite words on $A$ and $B \subset A$.

1. If $(W(B))_G$ (or $(W)_G$) is a complete collection of words on $B$, where $G = \{g(i)\}_{i=1}^{\infty}$, $r \geq 1$, and $g(i) \leq ri$ for all $i$, then $\text{ENT}(W, B, G) \geq \frac{\log |B|}{r}$. 

2. Conversely, if there exist $A, B \in A$ such that $\text{ENT}(W, \{A, B\}) > 0$, then there exists an increasing sequence $G = \{g(i)\}_{i=1}^{\infty}$ and $r \geq 1$ such that $g(i) \leq ri$ for all $i$ and $(W\{A, B\})_G$ is a complete collection on $\{A, B\}$.

**Theorem 8.** Suppose that $W$ is a closed shift-invariant collection of words on $A$, $G = \{g(i)\}_{i=1}^{\infty}$ is an increasing sequence of positive integers and $r > 1$ such that $g(i) < ri$. If $(W)_G$ is complete on $A$, then $\text{ ENT}(W, \{A, B\}) > 0$ for every distinct $A, B \in A$.

Suppose $W$ is a collection of words on $A$, $C \not\subset A$ and $A, B \in A$. For $W \in W$ define 
$$\phi_i(W) = \begin{cases} \pi_i(W) & \text{if } \pi_i(W) \not\in \{A, B\} \\ C & \text{if } \pi_i(W) \in \{A, B\} \end{cases}$$
and let $\phi(W) = (\phi_i(W))_{i=1}^{\infty}$. Then define $W/\{A, B\} = \{\phi(W)|W \in W\}$. If $W_n$ is a finite word of length $n$ then take $\phi(W_n) = (\phi_i(W_n))_{i=1}^{n}$.

The following 3 theorems are Corollary 17, Corollary 19 and Theorem 22 respectively in [4].

**Theorem 9.** Suppose that $W$ is a closed shift-invariant collection of words on $A$, $G = \{g(i)\}_{i=1}^{\infty}$ is an increasing sequence of positive integers and $r > 1$ such that $g(i) < ri$. If $\text{ENT}(W, \{A, B\}) = 0$, then $\text{ENT}(W, G) = \text{ENT}(W/\{A, B\}, G)$, $\text{ENT}((W)_G) = \text{ENT}((W/\{A, B\})_G)$ and hence $\text{ENT}(W) = \text{ENT}(W/\{A, B\})$.

**Theorem 10.** Suppose that $W$ is a closed shift-invariant collection of words on $A$, $G = \{g(i)\}_{i=1}^{\infty}$ is an increasing sequence of positive integers and $r > 1$ such that $g(i) < ri$. If
$$\text{ENT}((W)_G) > \log(|A| - 1)$$
then $\text{ENT}(W, \{A, B\}) > 0$ for every distinct $A, B \in A$. 


Theorem 11. Suppose that \( W \) is a closed shift-invariant collection of words on \( A \) and \( B \), and \( D \) are distinct elements of \( A \). If \( \text{ENT}(W, \{A, B, D\}) > 0 \) and \( \text{ENT}(W/\{A, B\}, \{C, D\}) > 0 \), where \( C \) \( (C \notin A) \) is the identification of \( A \) and \( B \) given by the previously defined \( \phi \), then \( \text{ENT}(W, \{A, D\}) > 0 \) or \( \text{ENT}(W, \{B, D\}) > 0 \).

Suppose that \( G = \{g(i)\}_{i=1}^{\infty} \) is an increasing sequence of positive integers. Let \( K_n = (k_1, \ldots, k_n) \) be a collection of not necessarily distinct positive integers. We say that \( G \) has pattern \( K_n \) if
\[
g(2i) - g(2i - 1) = k_1, \\
g(2(2i - 1) + 1) - g(2(2i - 1)) = k_2, \\
\vdots \\
g(2^{n-1}(2i - 1) + 1) - g(2^{n-1}(2i - 1)) = k_n
\]
for each positive integer \( i \). However, it is possible that \( g(2^ni + 1) - g(2^ni) \) is not constant as \( i \) varies.

The next theorem is Theorem 29 in [4].

Theorem 12. Suppose that \( W \) is a closed shift-invariant collection of words on \( A \) such that \( \text{ENT}(W) > 0 \). Then there exists \( A, B \in A \) such that
\[
\text{ENT}(W, \{A, B\}) > 0
\]
Furthermore, for each \( n \) there exists and increasing sequence of positive integers \( G(n) = \{g_n(i)\}_{i=1}^{\infty}, r_n > 1 \) and \( K_n = (k_1, \ldots, k_n) \) such that
1. \( \{W(\{A, B\})\}_{G(n)} \) is complete on \( \{A, B\} \).
2. \( G(n) \) has pattern \( K_n \).
3. \( g_n(i) \leq r_n i \) for each \( i \).

3. The Relationship between the Entropy of Words and the Entropy of Functions

In this section, we will use the structure of words to aid in finding results on the entropy of self maps. Let \( f : X \to X \) be an onto continuous function and let \( A \) be a finite open cover of \( X \). Cover \( A \) can be thought of as an alphabet where the elements of \( A \) are the symbols. Inverse images of \( f \) will then be used to create the words.

Let \( f : X \to X \) and \( A \) be an open cover on \( X \). A finite sequence \( \langle A_1, A_2, \ldots, A_i \rangle \) of elements \( A_k \in A \) is a \( (f, A) \)-word of length \( i \) provided
\[
A_1 \cap f^{-1}(A_2) \cap \ldots \cap f^{-i+1}(A_i) \neq \emptyset
\]

Let \( W_i(f, A) \) be the set of all \( (f, A) \)-words of length \( i \) and \( W(f, A) = \{W_i(f, A) \in W_i(f, A) \} \). The following proposition, although obvious, is very important.

Proposition 13. \( W(f, A) \) is a closed shift-invariant collection of words on \( A \).

Proof. The fact that \( W(f, A) \) is closed is obvious from the definition. To show that \( W(f, A) \) is shift invariant, let \( \langle A_i \rangle_{i=1}^{\infty} \in W(f, A) \). Then for each \( k \),
\[
A_1 \cap f^{-1}(A_2) \cap \ldots \cap f^{-k+1}(A_k) \neq \emptyset
\]
which implies that
\[
f(A_1) \cap A_2 \cap \ldots \cap f^{-k+2}(A_k) = f(A_1 \cap f^{-1}(A_2) \cap \ldots \cap f^{-k+1}(A_k)) \neq \emptyset.
\]
Hence,
\[ A_2 \cap f^{-1}(A_3) \cap \ldots \cap f^{-k+2}(A_k) \neq \emptyset. \]
Therefore, \( \langle A_i \rangle_{i=2}^k \in \mathcal{W}_{k-1}(f, \mathcal{A}) \) and it follows from the definition that
\[ \sigma(\langle A_i \rangle_{i=1}^\infty) = \langle A_i \rangle_{i=2}^\infty \in \mathcal{W}(f, \mathcal{A}) \]
\[ \square \]

The entropy on a collection of infinite words \( \mathcal{W}(f, \mathcal{A}) \) can now be defined as
\[ \text{ENT}(f, \mathcal{A}) = \text{ENT}(\mathcal{W}(f, \mathcal{A})) = \limsup_{i \to \infty} \frac{\log |\Pi_i(\mathcal{W}(f, \mathcal{A}))|}{i}. \]

The next two propositions compare the above definition of entropy \( \text{ENT}(f, \mathcal{A}) \) with the original definition of entropy \( \text{Ent}(f, \mathcal{A}) \) found in the first section.

**Proposition 14.** If \( \mathcal{A} \) is a finite open cover of \( X \) then
\[ |\mathcal{W}_k(f, \mathcal{A})| \geq N(\mathcal{V}^{k-1}_{i=0} f^{-i}(\mathcal{A})) \]

**Proof.** If \( \mathcal{V} \) is a subcover of \( \mathcal{V}^{k-1}_{i=0} f^{-i}(\mathcal{A}) \) such that \( |\mathcal{V}| = N(\mathcal{V}^{k-1}_{i=0} f^{-i}(\mathcal{A})) \) then for every \( V \in \mathcal{V} \) there exists a unique \( \langle A_1, A_2, \ldots, A_i \rangle \in \mathcal{W}_i(f, \mathcal{A}) \) such that \( V = A_1 \cap f^{-1}(A_2) \cap \ldots \cap f^{-k+1}(A_k) \). Therefore, the proposition holds. \[ \square \]

The next proposition follows immediately.

**Proposition 15.** \( \text{ENT}(\mathcal{W}(f, \mathcal{A})) \geq \text{Ent}(f, \mathcal{A}) \).

Next, suppose that \( \mathcal{B} \) is a finite collection of open sets that do not necessarily cover \( X \). Let \( G = \{g(i)\}_{i=1}^\infty \) be an increasing sequence of non-negative integers. Then define \( \langle B_i \rangle_{i=1}^k \) to be a \( (f, \mathcal{B}, G) \)-word of length \( k \) provided \( B_i \in \mathcal{B} \) and
\[ f^{-g(1)}(B_1) \cap f^{-g(2)}(B_2) \cap \ldots \cap f^{-g(k)}(B_k) \neq \emptyset. \]
Let \( \mathcal{W}_k(f, \mathcal{B}, G) \) be a collection of all \( (f, \mathcal{B}, G) \)-words of length \( k \). Then define \( W \in \mathcal{W}(f, \mathcal{B}, G) = (\mathcal{W}(f, \mathcal{B}))_G \) provided \( \Pi_k(W) \in \mathcal{W}_k(f, \mathcal{B}, G) \) for all \( k \). Let
\[ \text{ENT}(f, \mathcal{B}, G) = \limsup_{i \to \infty} \frac{\log |\Pi_i(\mathcal{W}(f, \mathcal{B}, G))|}{g(i)}, \]
and define the entropy of \( (f, \mathcal{B}) \) by
\[ \text{ENT}(f, \mathcal{B}) = \sup_G \{\text{ENT}(f, \mathcal{B}, G)\}. \]
Also, \( (\mathcal{W}(f, \mathcal{B}))_G \) and \( \mathcal{W}(f, \mathcal{B}, G) \) are themselves collections of words and
\[ \text{ENT}(\mathcal{W}(f, \mathcal{B}, G)) \geq \text{ENT}(\mathcal{W}(f, \mathcal{B}), G) = \text{ENT}(f, \mathcal{B}, G). \]

Due to the construction of the words, all of the theorems in section 2 can be applied by the substitutions:
\[ \mathcal{W}(f, \mathcal{A}) = \mathcal{W}(\mathcal{A}) \text{ or } \mathcal{W}, \]
\[ \mathcal{W}(f, \mathcal{B}, G) = (\mathcal{W}(\mathcal{B}))_G, \]
\[ \text{ENT}(f, \mathcal{A}) = \text{ENT}(\mathcal{W}(\mathcal{A})) \text{ and } \]
\[ \text{ENT}(f, \mathcal{B}, G) = \text{ENT}(\mathcal{W}(\mathcal{B}), G). \]

The next two propositions follow from the definitions.

**Proposition 16.** For any finite collection of open sets \( \mathcal{B} \) on \( X \), \( \text{ENT}(f, \mathcal{B}) \leq \log |\mathcal{B}| \)
Proposition 17. Suppose that \( B = \{B_1, ..., B_n\} \) and \( B' = \{B'_1, ..., B'_n\} \) are collections of open sets such that \( B_i \subset B'_i \) for each \( i \), then \( \text{ENT}(f, B) \leq \text{ENT}(f, B') \).

Theorem 18. Suppose that \( B \) is a finite collection of open sets and there exist \( A, B \in B \) such that \( \text{ENT}(f, \{A, B\}) = 0 \). Then \( \text{ENT}(f, (B\setminus \{A, B\}) \cup \{A \cup B\}) = \text{ENT}(f, B) \).

Proof. For every increasing sequence \( G \), it follows from the hypothesis that
\[
\text{ENT}(W(f, \{A, B\}), G) = \text{ENT}(f, \{A, B\}, G) = 0.
\]
Let \( C = A \cup B \). Then by Theorem 8, for each sequence \( G \)
\[
\text{ENT}(f, (B\setminus \{A, B\}) \cup \{A \cup B\}) = \text{ENT}(W(f, B)/\{A, B\}, G)
= \text{ENT}(W(f, B), G)
= \text{ENT}(f, B).
\]

Let \( K_n = (k_1, ..., k_n) \) be a collection of positive integers (not necessarily distinct).
Define \( R_{K_n}(B) \) recursively in the following way: Let \( D_0 = B \) and given \( D_{j-1} \) define \( D_j = D_{j-1} \vee f^{-k_j}(D_{j-1}) \). Let \( R_{K_n}(B) = D_\infty \). Notice that if \( B \) is disjoint, then \( R_{K_n}(B) \) must be disjoint. Also \( |R_{K_n}(B)| = |B|^{2^n} \).

The next result is the main theorem of this section:

Theorem 19. Suppose that \( A, B \) are open sets such that \( \text{ENT}(f, \{A, B\}) > 0 \). Then for each positive integer \( n \), there exists a finite set of positive integers \( K_n = \{k_1\}_{i=1}^n, r \geq 1, \) and \( G = \{g(i)\}_{i=1}^\infty \) such that \( g(i) \leq r_i \) and \( W(f, R_{K_n}(A, B), G) \) is complete on \( R_{K_n}(\{A, B\}) \).

Furthermore, for every distinct \( A', B' \in R_{K_n}(\{A, B\}) \), \( \text{ENT}(f, \{A', B'\}) > 0 \).

Proof. By Theorem 12, there exists an increasing sequence of positive integers \( G(n) = \{g_n(i)\}_{i=1}^\infty, r_n > 0 \) and \( K_n = (k_1, ..., k_n) \) such that \( G(n) \) has pattern \( K_n \) and \( g_n(i) \leq r_n i \). Let \( g(i) = g_n(2^n(i-1) + 1) \). Then
\[
U_1 \cap f^{-g(1)}(U_2) \cap ... \cap f^{-g(\ell-1)}(U_\ell) \neq \emptyset
\]
if and only if
\[
X_1 \cap f^{-g_1(1)}(X_2) \cap ... \cap f^{-g_\ell(\ell-1)}(X_\ell) \neq \emptyset
\]
where \( X_i \in \{A, B\} \) and
\[
U_i = \bigcap_{k=1}^{2^n} f^{-g_n(2^n(i-1)+k-1)}(X_2^{n(i-1)+k}) \in R_{K_n}(\{A, B\}).
\]

Thus, it follows that \( W(f, R_{K_n}(\{A, B\}), G) \) is complete on \( R_{K_n}(\{A, B\}) \) since \( W(f, \{A, B\}, G(n)) \) is complete on \( \{A, B\} \). Furthermore, if we let \( r = 2^{n+r} r_n \), then
\[
g(i) = g_n(2^n(i-1) + 1) \leq (2^n(i-1) + 1) r_n \leq 2^{n+1} r_n i = ri.
\]
Hence it follows from Theorem 8 that \( \text{ENT}(f, \{A', B'\}) > 0 \) for every distinct \( A', B' \in R_{K_n}(\{A, B\}) \).
4. Entropy and Open Covers

Recall that \( \mathcal{C} = \{C_1, C_2, ..., C_n\} \) is a chain cover of \( X \) provided \( C_i \cap C_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). \( \mathcal{C} \) is a proper chain cover provided \( C_i \cap C_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). The next proposition is well known and easy to show.

**Proposition 20.** If \( X \) is chainable, then for every \( \epsilon > 0 \) there exists a proper chain cover of \( X \) with mesh less that \( \epsilon \).

The goal of this section is to show that if \( \mathcal{C} \) is a chain cover such that \( \text{ENT}(f, \mathcal{C}) > \log(2) \), then there exist disjoint open sets \( A, B \) such that \( \text{ENT}(f, \{A, B\}) > 0 \).

If \( \mathcal{U} \) is a collection of sets, define the star of \( \mathcal{U} \) as \( \mathcal{U}^* = \bigcup_{U \in \mathcal{U}} U \).

**Lemma 21.** Suppose that \( G = \{g(i)\}_{i=1}^\infty \) is a sequence of positive integers, \( r \geq 1 \) such that \( g(i) \leq ri \) and \( \text{ENT}(W(f, B, G)) > \log(|B| - 1) \). Let \( H \in B \) and \( \mathcal{U}(H) \) be a finite collection of open sets such that \( H \subset \mathcal{U}(H)^* \), \( \mathcal{U}(H) \cap (B - \{H\}) = \emptyset \) and \( |B| > |\mathcal{U}(H)| \). Then there exists \( A \in \mathcal{U}(H) \) and a \( B' \subset B - \{H\} \) such that \( |B'| \geq |B| - |\mathcal{U}(H)| \) and \( \text{ENT}(f, \{A, B'\}) > 0 \) for every \( B \in B' \).

**Proof.** Let \( \mathcal{U}(H) = \{U_1, U_2, ..., U_n\} \) and suppose that the lemma is false for \( \{U_1, U_2, ..., U_n\} \). Then there exist distinct \( C_1, ..., C_{n-1} \in B - \{H\} \) such that \( \text{ENT}(f, \{U_i, C_i\}) = 0 \) for each \( 1 \leq i \leq n - 1 \). Let

\[
\mathcal{B}_i = B - \{H, C_1, ..., C_i\},
\]

and

\[
\mathcal{D}_i = B_i \cup \{U_1 \cup C_1, ..., U_i \cup C_i, U_{i+1}, ..., U_n\}.
\]

**Claim** \( \text{ENT}(W(f, (B - \{H\}) \cup \mathcal{U}(H), G)) \geq \text{ENT}(W(f, B, G)) \).

Let \( W_n \) be a \( (f, B, G) \)-word of length \( n \) and let \( \{q_{n,j}\}_{j=1}^{p_n} \subset \{1, ..., n\} \) such that \( \pi_{q_{n,j}}(W_n) = H \). Then there exists \( \{U_j\}_{j=1}^{p_n} \) such that \( U_j \in \mathcal{U}(H) \) and a \( (f, (B - \{H\}) \cup \mathcal{U}(H), G) \)-word \( Q_n \) of length \( n \) such that

\[
\pi_i(Q_n) = \begin{cases} 
\pi_i(W_n) & \text{if } i \notin \{q_{n,j}\}_{j=1}^{p_n} \\
U_j & \text{if } i = q_{n,j}.
\end{cases}
\]

Then let

\[
\Psi_n : \Pi_n(W(f, (B - \{H\}) \cup \mathcal{U}(H), G)) \rightarrow \Pi_n(W(f, B, G))
\]

be defined by

\[
\pi_i(\Psi(\langle A_i \rangle_{i=1}^\infty)) = \begin{cases} 
A_i & \text{if } A_i \in B - \{H\} \\
H & \text{if } A_i \in \mathcal{U}(H).
\end{cases}
\]

Then \( \Psi_n(Q_n) = W_n \) in the previous definitions of \( Q_n \) and \( W_n \). Therefore, \( \Psi_n \) is onto so

\[
|\Pi_n(W(f, (B - \{H\}) \cup \mathcal{U}(H), G))| \geq |\Pi_n(W(f, B, G))|
\]

and the claim now follows.

Next, because \( U_1, C_1 \in (B - \{H\}) \cup \mathcal{U}(H) \) and \( \text{ENT}(W(f, \{U_1, C_1\})) = 0 \), it follows from Theorem 9 that

\[
\text{ENT}(W(f, \mathcal{D}_1, G)) = \text{ENT}(W(f, (B - \{H\}) \cup \mathcal{U}(H), G)),
\]

where it is assumed that \( A = U_1, B = C_1 \) and \( C = U_1 \cup C_1 \).
Likewise, since \( U_j, C_j \in \mathcal{D}_{j-1} \) and \( \text{ENT}(f, \{ U_j, C_j \}) = 0 \), it follows from Theorem 9 (recall that from Proposition 13, \( \mathcal{W}(f, \mathcal{D}_j) \) must be a closed shift-invariant collection of words on \( \mathcal{D}_j \)) that
\[
\text{ENT}(\mathcal{W}(f, \mathcal{D}_j, G)) = \text{ENT}(\mathcal{W}(f, \mathcal{D}_{j-1}, G)).
\]
Hence,
\[
\text{ENT}(\mathcal{W}(f, \mathcal{B}_{n-1} \cup \{ U_1 \cup \ldots \cup U_{n-1} \cup C_{n-1}, U_n \}, G)) = \text{ENT}(\mathcal{W}(f, \mathcal{B}_{n-1}, G))
\geq \text{ENT}(\mathcal{W}(f, \mathcal{B}, G))
> \log(|\mathcal{B}| - 1).
\]
Therefore, since \(|\mathcal{B}_{n-1} \cup \{ U_1 \cup \ldots \cup U_{n-1} \cup C_{n-1}, U_n \}| = |\mathcal{B}|\), we can use Theorem 10 to conclude that \( \text{ENT}(f, \{ A, B \}) > 0 \) for every distinct \( A, B \in \mathcal{B}_{n-1} \cup \{ U_1 \cup C_1, \ldots, U_{n-1} \cup C_{n-1}, U_n \} \). Thus, if we let \( B' = \mathcal{B}_{n-1} \), then \( \text{ENT}(f, \{ U_n, B \}) > 0 \) for each \( B \in B' \) and \(|B'| = |\mathcal{B}| - n \geq |\mathcal{B}| - |\mathcal{U}(H)|\).

\[\square\]

\textbf{Lemma 22.} Suppose that \( \mathcal{B} \) is a disjoint collection of open sets, \( G = \{ g(i) \}_{i=1}^{\infty} \) is an increasing sequence of positive integers and \( r \geq 1 \) such that \( g(i) \leq ri \) and \( \text{ENT}(\mathcal{W}(f, \mathcal{B}, G)) > \log(|\mathcal{B}| - 1) \). Let \( H \in \mathcal{B} \), \( \mathcal{U}(H) \) be a collection of open sets, and \( \{ B_i \}_{i=1}^{m} \) be a collection of subsets of \( \mathcal{B} \) such that:

1) \( H \subset \mathcal{U}(H)^* \)
2) \( B_i \subset \mathcal{B} - \{ H \} \)
3) \(|B_i| > |\mathcal{U}(H)|\)
4) For each \( U \in \mathcal{U}(H) \) and \( B \in B_i \), \( U \cap B = \emptyset \).

Then there exists \( U \in \mathcal{U}(H) \) such that for each \( i \in \{ 1, \ldots, m \} \), there exists \( B_i \in B_i \) such that \( \text{ENT}(f, \{ U, B_i \}) > 0 \).

\textbf{Proof.} By Lemma 21, there exists \( B' \subset \mathcal{B} - \{ H \} \) and \( U \in \mathcal{U}(H) \) such that for each \( B \in B' \), \( \text{ENT}(f, \{ U, B \}) > 0 \) and \(|B'| \geq |\mathcal{B}| - |\mathcal{U}(H)|\). Since for each \( i \), \( B' \cup B_i \subset \mathcal{B} \) and
\[
|B'| + |B_i| > |B'| + |\mathcal{U}(H)| \geq |B|,
\]
it follows that \( B' \cap B_i \neq \emptyset \). So if \( B_i \in B' \cap B_i \), then \( \text{ENT}(f, \{ U, B_i \}) > 0 \).

\[\square\]

\textbf{Theorem 23.} Suppose that \( A, B \) are elements in cover \( \mathcal{B} \) such that \( A \cap B = \emptyset \) and \( \text{ENT}(f, \{ A, B \}) > 0 \). If \( B \subset C \cup D \), where \( C \) and \( D \) are open sets, then \( \text{ENT}(f, \{ A, C \}) > 0 \) or \( \text{ENT}(f, \{ A, D \}) > 0 \) where \( C \) and \( D \) are open sets.

\textbf{Proof.} Since \( \text{ENT}(f, \{ A, B \}) > 0 \) and \( B \subset C \cup D \), it follows that \( \text{ENT}(f, \{ A, C, D \}) > 0 \). Hence the theorem follows from Theorem 11.

\[\square\]

\textbf{Corollary 24.} Suppose \( \text{ENT}(f, \{ A, B \}) > 0 \), \( \mathcal{B} \) is a collection of open sets such that \( B \subset \mathcal{B}^* \). Then there exists \( B' \in \mathcal{B} \) such that \( \text{ENT}(f, \{ A, B' \}) > 0 \).

\textbf{Proof.} Let \( \mathcal{B} = \{ B_1, \ldots, B_n \} \). Proof is by induction on \( n \).

\textbf{Base Case.} Let \( n = 2 \), then the corollary follows from Theorem 23.

\textbf{Induction Step.} Suppose that the corollary is true for \( n = k \) and let \( \mathcal{B}_{k+1} = \{ B_1, \ldots, B_{k+1} \} \) be a collection of open sets such that \( B \subset \mathcal{B}_{k+1}^* \). If \( \text{ENT}(f, \{ A, B_{k+1} \}) > 0 \) then we are done. So suppose that \( \text{ENT}(f, \{ A, B_{k+1} \}) = 0 \) and let \( \mathcal{B}_k = \)
\{B_1, \ldots, B_k\}. Then clearly \(B \subset B_k \cup \{B_{k+1}\}\) so it follows from Theorem 23 that \(\text{ENT}(f, \{A, B_i\}) > 0\). Then by the induction hypothesis, it follows that there exist \(B_i \in B_k\) such that \(\text{ENT}(f, \{A, B_i\}) > 0\). \(\Box\)

**Corollary 25.** Suppose \(\text{ENT}(f, \{A, B\}) > 0\), \(A, B\) are collections of open sets such that \(A \subset A^*\) and \(B \subset B^*\). Then there exists \(A' \in A\) and \(B' \in B\) such that \(\text{ENT}(f, \{A', B'\}) > 0\).

**Proof.** Since \(A \subset A^*\) and \(B \subset B^*\) it follows that \(\text{ENT}(f, \{A^*, B^*\}) > 0\). Hence by Corollary 24, there exists \(A' \in A^*\) such that \(\text{ENT}(f, \{A', B^*\}) > 0\). But then by another application of Corollary 24, there exists \(B' \in B^*\) such that \(\text{ENT}(f, \{A', B'\}) > 0\). \(\Box\)

**Lemma 26.** Suppose that \(f : X \to X\) and that \(A, B\) are open sets such that \(A \cap B = \emptyset\) and \(\text{ENT}(f, \{A, B\}) > 0\). Then for every positive integer \(m\), there exists a \(\delta > 0\) such that if \(U\) is a finite open cover of \(X\) with \(\text{mesh}(U) < \delta\), then \(U\) has the following properties:

1) There exist an integer \(p(m)\) and a sequence of positive integers \(K_{p(m)} = \{k_1, \ldots, k_{p(m)}\}\) such that \(|R_{K_{p(m)}}(\{A, B\})| \geq m + 1\).

2) There exists disjoint \(\{V, U_1, \ldots, U_m\} \subset U\) such that \(\text{ENT}(f, \{V, U_i\}) > 0\) for each \(i\).

3) Each element of \(\{V, U_1, \ldots, U_m\}\) intersects a unique element of \(R_{K_{p(m)}}(\{A, B\})\).

**Proof.** Let \(p(m)\) be such that \(m + 1 \leq 2^{p(m)}\). By Theorem 19, there exist an increasing sequence of positive integers \(G = \{g(i)\}_{i=1}^{\infty}, r \geq 1\) and a set of (not necessarily distinct) positive integers \(K_{p(m)} = \{k_1, \ldots, k_{p(m)}\}\) such that \(W(f, R_{K_{p(m)}}(\{A, B\}), G)\) is complete on \(R_{K_{p(m)}}(\{A, B\})\), where \(g(i) \leq r_i\).

Since \(A \cap B = \emptyset\), it follows that if \(A', B'\) are distinct elements of \(R_{K_{p(m)}}(\{A, B\})\), then \(\overline{A'} \cap B' = \emptyset\) and \(\overline{A'} \cap B' = \emptyset\). So choose \(\delta > 0\) and such that

\[
\delta < \frac{1}{3} \min \{d(A', B')|A' \neq B'\text{ and }A', B' \in R_{K_{p(m)}}(\{A, B\})\}.
\]

Let \(U\) be a chain cover of \(X\) with mesh less than \(\delta\). Notice that \(|R_{K_{p(m)}}(\{A, B\})| = 2^{p(m)} \geq m + 1\), so pick any set of distinct elements \(\{H, D_1, \ldots, D_m\}\) of \(R_{K_{p(m)}}(\{A, B\})\). Then define

\[
U(H) = \{U \in U|U \cap H \neq \emptyset\}
\]

and

\[
U(D_i) = \{U \in U|U \cap D_i \neq \emptyset\}.
\]

Let \(n = |U(H)|\) and find \(\alpha > p(m)\) such that \(n + 1 \leq 2^\alpha / 2^{2p(m)}\). By Theorem 19, there exist positive integers \(K_{\alpha} = \{k_1, k_2, \ldots, k_\alpha\}\) (where the first \(p(m)\) terms are the same as before), a subsequence \(G' = \{g'(i)\}_{i=1}^{\infty}\) of \(G\), and \(r' \geq 1\) such that \(g'(i) \leq r' i\) and \(W(f, R_{K_{\alpha}}(\{A, B\}), G')\) is complete on \(R_{K_{\alpha}}(\{A, B\})\).

Next let

\[
Q(H) = \{Q \in R_{K_{\alpha}}(\{A, B\})|Q \subset H\}
\]

and

\[
Q(D_i) = \{Q \in R_{K_{\alpha}}(\{A, B\})|Q \subset D_i\}.
\]

Then

\[
|Q(H)| = |Q(D_i)| = 2^{2^\alpha / 2^{2p(m)}} \geq n + 1.
\]
Let \( Q_H \) be any element of \( Q(H) \) and \( U(Q_H) = \{ U \cap Q_H | U \in \mathcal{U}(H) \} \). Then \( |U(Q_H)| \leq |\mathcal{U}(H)| = n \). Also, since \( \mathcal{W}(f, R_{K_n}([A,B]), G') \) is complete on \( R_{K_n}([A,B]) \), it follows that

\[
\text{ENT}(\mathcal{W}(f, R_{K_n}([A,B]), G')) = \limsup_{i \to \infty} \frac{\log(|R_{K_n}([A,B])|)}{i} = \log(|R_{K_n}([A,B])|).
\]

Furthermore, the following are all true

1) \( Q_H \subset \mathcal{U}(Q_H)^* \)
2) \( Q(D_i) \subset R_{K_n}([A,B]) - \{ Q_H \} \)
3) \( |Q(D_i)| \geq n + 1 > |\mathcal{U}(Q_H)| \)
4) For each \( U_{Q_H} \in \mathcal{U}(Q_H) \) and \( Q_{D_i} \in Q(V_i), U_{Q_H} \cap Q_{D_i} = \emptyset \).

Therefore, by Lemma 22, there exists \( V' \in \mathcal{U}(Q_H) \) such that for each \( i \) there exists \( Q_{D_i} \in Q(D_i) \) such that \( \text{ENT}(f, \{ V', Q_{D_i} \}) > 0 \). Consequently, \( Q_{D_i} \subset D_i \) and there exists a \( V \in \mathcal{U}(H) \) such that \( V' \subset V \). So by Proposition 17, \( \text{ENT}(f, \{ V, D_i \}) > 0 \) for each \( i \). Additionally, it follows from Corollary 24 that there exists \( U_i \in \mathcal{U}(D_i) \) such that \( \text{ENT}(f, \{ V, U_i \}) > 0 \).

It follows from the fact that mesh(\( \mathcal{U} \)) < \( \delta \) that \( \{ V, U_1, ..., U_m \} \) are all disjoint.

The next theorem gives a result about chain covers.

**Theorem 27.** If \( \mathcal{C} \) is a chain cover continuum such that \( \text{ENT}(f, \mathcal{C}) > \log(2) \), then there exist disjoint open sets \( C_i, C_j \) of \( \mathcal{C} \) such that \( \text{ENT}(f, \{ C_i, C_j \}) > 0 \).

**Proof.** The proof is by induction on the number of elements \( n \) in \( \mathcal{C} \). Since \( \text{ENT}(f, \mathcal{C}) > \log(2) \), we can assume that \( n \geq 3 \).

**Base Case.** Suppose \( n = 3 \), then by Theorem 10, \( \text{ENT}(f, \{ C_1, C_3 \}) > 0 \).

**Induction Step.** Suppose the theorem is true for \( n = k \) and let \( \mathcal{C}_{k+1} = \{ C_1, ..., C_{k+1} \} \) be a chain cover. If \( \text{ENT}(f, \{ C_1, C_3 \}) > 0 \), then we are done; so suppose that \( \text{ENT}(f, \{ C_1, C_3 \}) = 0 \). Let \( \mathcal{C}'_k = \{ C'_1, ..., C'_k \} \) where \( C'_1 = C_2, C'_2 = C_1 \cup C_3 \) and \( C'_i = C_{i+1} \) for \( i \geq 3 \). Then \( \mathcal{C}'_k \) is a proper chain cover with \( k \) elements and by Theorem 9, \( \text{ENT}(f, \mathcal{C}'_k) > \log(2) \). Hence, from the induction hypothesis, there exist disjoint \( C'_i, C'_j \in \mathcal{C}'_k \) such that \( \text{ENT}(f, \{ C'_i, C'_j \}) > 0 \). Suppose that \( i < j \); we have 3 cases:

**Case 1.** Suppose that \( i = 1 \). Then since \( C'_i \) and \( C'_j \) are disjoint, \( j \geq 3 \). Hence

\[
\text{ENT}(f, \{ C_2, C_{j+1} \}) = \text{ENT}(f, \{ C'_1, C'_j \}) > 0.
\]

Since \( j + 1 \geq 4 \), \( C_2 \) and \( C_{j+1} \) must be disjoint and the theorem follows.

**Case 2.** Suppose that \( i = 2 \). Then since \( C'_i \) and \( C'_j \) are disjoint, \( j \geq 4 \). Hence

\[
\text{ENT}(f, \{ C_1 \cup C_3, C_{j+1} \}) = \text{ENT}(f, \{ C'_2, C'_j \}) > 0.
\]

Then by Theorem 23, one of the following must be true:

\[
\text{ENT}(f, \{ C_1, C_{j+1} \}) > 0 \text{ or } \text{ENT}(f, \{ C_3, C_{j+1} \}) > 0.
\]
Suppose on the contrary that for each \( i \) such
\[ \pi \leq \quad \leq \]
or
But then that implies that one of the following, respectively, is true:
\[ \{ \text{Have } i \text{ exists an } \} \]
\[ \text{Proof.} \]
\[ \text{Since } j \text{ is a chainable continuum to be indecomposable it must consist of nested crooked chains. Let } C_0 \text{ and } C_1 = \{ C_1^1, C_1^2, ..., C_1^n \} \text{ be chains such that } C_1 \text{ refines } C_0. \]
\[ \text{If } A \text{ and } B \text{ are disjoint elements of } C_0, \text{ then } C_1 \text{ is crooked between } A \text{ and } B \text{ if there exist } i < j < k \text{ such that either} \]
\[ \begin{align*}
1) & \quad C_1^i \cap A \neq \emptyset, C_1^k \cap A \neq \emptyset, \text{ and } C_1^j \cap B \neq \emptyset. \\
2) & \quad C_1^i \cap B \neq \emptyset, C_1^k \cap B \neq \emptyset, \text{ and } C_1^j \cap A \neq \emptyset.
\end{align*} \]
The next theorem gives the connection between crookedness and indecomposablity and can be found in [1] in a slightly different form.

**Theorem 28.** Let \( C_0 \) be a chain cover of continuum \( X \), \( A_0, B_0 \in C_0 \) such that \( A_0 \cap B_0 = \emptyset \) and \( \epsilon = \frac{1}{4}d(A, B) \). If for each \( i \) there exists a chain-cover \( C_i \) and \( A_i, B_i \in C_{i+1} \) such that
\[ \begin{align*}
1) & \quad \text{mesh}(C_i) < \epsilon/3^i. \\
2) & \quad C_i \text{ closure refines } C_{i-1}. \\
3) & \quad \text{The subchain of } C_i \text{ between } A_i \text{ and } B_i \text{ is crooked between } A_{i-1} \text{ and } B_{i-1}.
\end{align*} \]
Then \( X = \bigcap_{i=0}^{\infty} C_i \) contains an non-degenerate indecomposable subcontinuum.

The next 2 technical lemmas that will aid in proving the main result.

**Lemma 29.** Let \( W_1 = \{ \langle U_i \rangle_{i=1}^6 \mid U_i \in \{ A, B \} \} \) and suppose that \( W_1, ..., W_6 \) are 6 distinct words of \( W_4 \). Then there exists \( i \in \{ 1, ..., 4 \} \) and \( 1 < j < k \) such that \( \pi_i(W_{j}) = \pi_i(W_{k}) \) and \( \pi_i(W_{j}) \neq \pi_i(W_{k}) \).

**Proof.** Since the words are comprised only 2 symbols, it suffices to show that there exists an \( i' \) and \( 1 < j < k \) such that \( \pi_i'(W_1) \neq \pi_i'(W_j) \) and \( \pi_i'(W_k) \neq \pi_i'(W_{i'}) \). Suppose on the contrary that for each \( i \) there is at most 1 integer \( n_i \in \{ 1, ..., 6 \} \) such \( \pi_i(W_{n_i}) \neq \pi_i(W_{n_i+1}) \). Let
\[ \Psi(i) = \begin{cases} 
 n_i & \text{if } \pi_i(W_{n_i}) \neq \pi_i(W_{n_i+1}) \\
 0 & \text{if } \pi_i(W_j) = \pi_i(W_k) \text{ for every } j, k \in \{ 1, ..., 6 \}.
\end{cases} \]
Have \( \{ i_k \}_{k=1}^{\infty} \) be a finite sequence such that \( \Psi(i_k) \leq \Psi(i_{k+1}) \) for all \( k \). Notice that \( 0 \leq \Psi(i_k) \leq 5 \). Then by the pigeon-hole principle one of the following must be true:
\[ \begin{align*}
1) & \quad \Psi(i_1) \geq 2 \\
2) & \quad \text{There must be a } k \text{ such that } \Psi(i_{k+1}) - \Psi(i_k) \geq 2. \\
3) & \quad \Psi(i_k) \leq 4.
\end{align*} \]
But then that implies that one of the following, respectively, is true:
\[ \begin{align*}
1) & \quad W_1 = W_2 \\
2) & \quad W_{\Psi(i_{k+1}) - 1} = W_{\Psi(i_{k+1})}
\end{align*} \]
Lemma 30. Let \( h : X \to X \) be a homeomorphism, \( A, B \) be open sets such that \( A \cap B = \emptyset \), and \( \mathcal{U} = \{U_1, ..., U_{p_1}, ..., U_{p_2}, ..., U_{p_n}, ..., U_m \} \) be a chain cover of \( X \) such that \( \{U_{p_i}\}_{i=1}^n \) all intersect unique elements of \( R_{K_2}(\{A, B\}) \) where \( K_2 = \{k_1, k_2\} \). Then there exist \( 1 < i < k \leq 6 \) and \( n \in \{0, k_1, k_2, k_1 + k_2\} \) such that either

1) \( h^n(U_{p_i}) \cap B \neq \emptyset \), \( h^n(U_{p_k}) \cap B \neq \emptyset \), and \( h^n(U_{p_i}) \cap A \neq \emptyset \) or
2) \( h^n(U_{p_i}) \cap A \neq \emptyset \), \( h^n(U_{p_k}) \cap A \neq \emptyset \), and \( h^n(U_{p_i}) \cap B \neq \emptyset \).

Proof. Let \( W_j = \langle Y_1^j, Y_2^j, Y_3^j, Y_4^j \rangle \) be the word associated with the unique element of \( \mathcal{U}_j \) that intersects \( U_{p_i} \), where \( Y_i^j \in \{A, B\} \). Then by Lemma 29, there exists \( 1 < j < k \) and \( i' \in \{1, ..., 4\} \) such that \( Y_i^j = Y_i^{k'} \) and \( Y_i^j \neq Y_i^{k'} \).

First, suppose that \( Y_i^j = Y_i^{k'} = B \) and \( Y_i^{k'} = A \). Then using the value of \( n_i' \) defined by \( n_1 = 0, n_2 = k_1, n_3 = k_2 \) and \( n_4 = k_1 + k_2 \), it follows that

\[
U_{p_i} \cap h^{n_i'}(B) \neq \emptyset, U_{p_k} \cap h^{n_i'}(B) \neq \emptyset \quad \text{and} \quad U_{p_j} \cap h^{n_i'}(A) \neq \emptyset,
\]

so 1) is true. On the other hand, if \( Y_i^j = Y_i^{k'} = A \) and \( Y_i^{k'} = B \), then 2) follows in a similar manner. \( \Box \)

The next lemma shows the relationship between entropy and crookedness.

Lemma 31. Suppose that \( h : X \to X \) is a homeomorphism of a chainable continuum and \( A, B \) are disjoint elements of proper chain cover \( C \) such that \( \text{ENT}(h, \{A, B\}) > 0 \). Then for every \( \epsilon > 0 \), there exists a chain cover \( \mathcal{U} \) and elements \( U, V \in \mathcal{U} \) with the following properties:

1) \( \text{mesh}(\mathcal{U}) < \epsilon \).
2) \( \text{Ent}(h, \{U, V\}) > 0 \).
3) The subchain from \( U \) to \( V \) is crooked between \( A \) and \( B \).

Proof. Let \( \epsilon > 0, m = 9 \) and \( p(m) = 2 \). By Theorem 19, there exist an increasing sequence of positive integers \( G = \{g(i)\}_{i=1}^\infty \), \( r \geq 1 \) and positive integers \( k_1, k_2 \) such that \( \mathcal{W}(h, R_{K_2}(\{A, B\}), G) \) is complete on \( R_{K_2}(\{A, B\}) \), where \( K_2 = \{k_1, k_2\} \), and \( g(i) \leq ri \). By uniform continuity, there exists a \( \delta > 0 \) such that if \( d(x, y) < \delta \) then \( d(h^n(x), h^n(y)) < \epsilon \) for all \( -k_1 - k_2 \leq n \leq k_1 + k_2 \). Let \( \mathcal{U} \) be a chain cover of \( X \) with mesh less than \( \delta \). Then by Lemma 26, we can assume that there exist distinct elements \( \{V, U_1, ..., U_9\} \) of \( \mathcal{U} \) such that each element intersects a unique element of \( R_{K_2}(\{A, B\}) \) and \( \text{ENT}(h, \{U, V\}) > 0 \) for each \( i \).

In the ordering of chain \( \mathcal{U} \), at least 5 elements of \( \{U_1, ..., U_9\} \) either follow \( V \) or precede \( V \). Since each element of \( R_{K_2}(\{A, B\}) \) is of the form \( Y_i \cap h^{-k_1}(Y_2) \cap h^{-k_2}(Y_3) \cap h^{-k_1-k_2}(Y_4) \) where \( Y_i \in \{A, B\} \), it follows from Lemma 41 that there exists \( j \in \{0, k_0, k_1, k_0 + k_1\} \) and \( U_a, U_b \) such that

1) \( U_a \) is between \( V \) and \( U_b \) in the ordering of \( \mathcal{U} \) and either
2) \( h^j(V) \cap B \neq \emptyset, h^j(U_a) \cap B \neq \emptyset \), and \( h^j(U_a) \cap A \neq \emptyset \) or
\[ h^j(V) \cap A \neq \emptyset, \ h^j(U_b) \cap A \neq \emptyset, \text{ and } h^j(U_a) \cap B \neq \emptyset. \]

Hence, the lemma follows. \[ \square \]

The next Theorem is the main result of this paper.

**Theorem 32.** Suppose that \( X \) is a chainable continuum and \( h : X \rightarrow X \) is a homeomorphism such that \( \text{Ent}(h) > 0 \). Then \( X \) must contain an indecomposable subcontinuum.

**Proof.** By Proposition 2, we know that there exists a homeomorphism \( h \) such that \( \text{Ent}(h) > \log(3) \). Then by Propositions 1 and 3 and the fact that \( \text{Ent}(h) \) can be estimated by the entropy on finite cover, there exists a proper chain cover \( C_0 \) of \( X \) such that \( \text{ENT}(h, C_0) > \log(3) \). Then by Theorem 27, there exist \( A, B \in C_0 \) such that \( A \cap B = \emptyset \) and \( \text{ENT}(h, \{A, B\}) > 0 \). Let \( \gamma < \frac{1}{3} \text{d}(A, B) \).

Continuing inductively suppose that a chain cover \( C_i \) has been found with the following properties:
1) \( \text{mesh}(C_i) < \gamma/3^i \)
2) There exist \( A_i, B_i \in C_i \) such that \( A_i \cap B_i = \emptyset \) and \( \text{ENT}(h, \{A_i, B_i\}) > 0 \).

Then by Lemma 31, there exists a chain-cover \( C_{i+1} \) such that
1) \( C_{i+1} \) closure refines \( C_i \).
2) \( \text{mesh}(C_{i+1}) < \gamma/3^{i+1} \)
3) There exist \( A_{i+1}, B_{i+1} \in C_{i+1} \) such that \( A_{i+1} \cap B_{i+1} = \emptyset \) and \( \text{ENT}(h, \{A_{i+1}, B_{i+1}\}) > 0 \).
4) The subchain of \( C_{i+1} \) between \( A_{i+1} \) and \( B_{i+1} \) is crooked between \( A_i \) and \( B_i \).

Since \( X = \bigcap_{i=-\infty}^{\infty} C_i \), it follows from Theorem 28 that \( X \) contains a nondegenerate indecomposable subcontinuum. \[ \square \]

A function \( f \) is open if \( f(U) \) is open for each open set \( U \) in the domain of \( f \). A function \( f \) is monotone if \( f^{-1}(H) \) is connected for every connected \( H \) in the range of \( f \). Homeomorphisms are monotone and open. So the following are natural questions:

**Question 1:** If \( f : X \rightarrow X \) is a monotone continuous function such that \( \text{Ent}(f) > 0 \), then must \( X \) contain a non-degenerate indecomposable subcontinuum?

If so, then consider the next question.

**Question 2:** If \( f : X \rightarrow X \) is a monotone continuous function such that \( \text{Ent}(f) > 0 \), then must \( X \) contain a non-degenerate indecomposable subcontinuum?

However, it is well known that the tent map \( f : [0, 1] \rightarrow [0, 1] \) defined by

\[
 f(x) = \begin{cases} 
 2x & \text{if } x \in [0, 1/2] \\
 2 - 2x & \text{if } x \in (1/2, 1] 
\end{cases}
\]

in an open continuous function such that \( \text{Ent}(f) = \log(2) \). Also, the cone over the Cantor set (also known as the Cantor Fan) admits a positive entropy homeomorphism. The cone over the Cantor set is tree-like but not chainable and does not contain a non-degenerate indecomposable subcontinuum.
6. References


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