EXPANSIVE HOMEOMORPHISMS AND PLANE SEPARATING CONTINUA

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Abstract. A homeomorphism $h : X \to X$ is expansive provided that for some fixed $c > 0$ and every $x, y \in X$ there exists an integer $n$, dependent only on $x$ and $y$, such that $d(h^n(x), h^n(y)) > c$. It is shown that if $X$ is a 1-dimensional continuum that separates the plane into 2 pieces, then $h$ cannot be expansive.

1. Introduction

A continuum is a compact, connected metric space. A continuum is a plane continuum if it can be embedded in the plane. A homeomorphism $h : X \to X$ is called expansive provided for some fixed $c > 0$ and every $x, y \in X$ there exists an integer $n$ such that $d(h^n(x), h^n(y)) > c$. Expansive homeomorphisms exhibit sensitive dependence on initial conditions in the strongest sense in that no matter how close any two points are, their images will eventually be a certain distance apart.

One problem of interest is the classification of plane continua that admit (or do not admit) expansive homeomorphisms. The Plykin attractor [6] is a 1-dimensional plane continuum that admits an expansive homeomorphism. A 2-dimensional plane continuum that admits an expansive homeomorphisms has been constructed in [5]. It is known that tree-like continua and hence, 1-dimensional non-separating plane continua, do not admit expansive homeomorphisms [4]. A $n$-separating plane continuum separates the plane into $n$ complementary domains. The Plykin attractor is a 4-separating plane continuum. The main result of this paper will show that 1-dimensional 2-separating plane continua do not admit expansive homeomorphisms. The result is still unknown for 3-separating plane continua.

In order for a homeomorphism to be expansive, stretching of subcontinua by the homeomorphism must occur. In compact spaces, this means that subcontinua must either be stretched and wrapped or stretched and folded. The dyadic solenoid [8] and the Plykin attractor are examples of continua that admit expansive homeomorphisms that wrap subcontinua. On the other hand, when subcontinua are stretched and folded, some points do move closer together. Under this type of action, it appears that the homeomorphism will not be expansive. This is evident in the result that tree-like continua do not admit expansive homeomorphisms. Similarly, homeomorphisms that stretch subcontinua of 2-separating plane continua must have some bending. It from this fact that the main result will be shown.

2000 Mathematics Subject Classification. Primary: 54H20, 54F50, Secondary: 54E40.

Key words and phrases. expansive homeomorphism, 1-cyclic continuum, plane separating continuum.
2. Characterization of 1-dimensional 2-separating plane continua

We begin with several important definitions. Let $\mathcal{U}$ be an open cover. The mesh of $\mathcal{U}$ is then defined by

$$\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) | U \in \mathcal{U}\}.$$ 

For $U \in \mathcal{U}$, the core of $U$ is defined as

$$\text{core}(U) = \bigcap \{U - V | V \in \mathcal{U} - \{U\}\}.$$ 

A cover is taut if $U \cap V = \emptyset$ for all disjoint $U, V \in \mathcal{U}$. From here on out, we will assume that all covers are taut. A cover $\mathcal{V}$ refines $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$; $\mathcal{V}$ closure refines $\mathcal{U}$ if $\overline{\mathcal{V}} \subset \mathcal{U}$; and $\mathcal{V}$ 2-refines $\mathcal{U}$ if for every $V_i, V_j \in \mathcal{V}$ such that $V_i \cap V_j \neq \emptyset$, there exists a $U \in \mathcal{U}$ such that $V_i \cup V_j \subset U$. $\mathcal{U}$ is an amalgamation of $\mathcal{U}'$ if each element of $\mathcal{U}'$ is the union of elements of $\mathcal{U}$. Define the star of $\mathcal{U}$ as

$$\mathcal{U}^* = \bigcup_{U \in \mathcal{U}} U.$$ 

A chain $[C_1, C_2, \ldots, C_n]$ is a collection of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A circle-chain $[C_1, C_2, \ldots, C_n]$ is a collection of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ or $|i - j| = n - 1$. A collection of open sets $\mathcal{U}$ is connected if for every $U, U' \in \mathcal{U}$, there exists a chain from $U$ to $U'$ in $\mathcal{U}$.

The nerve of cover $\mathcal{U}$, denoted $\mathcal{N}(\mathcal{U})$, is a geometric simplex (graph) where each element $U_i \in \mathcal{U}$ is represented by a vertex $v_i \in \mathcal{N}(\mathcal{U})$ and there exists an arc (edge) in $\mathcal{N}(\mathcal{U})$ from $v_i$ to $v_j$ if and only if $U_i \cap U_j \neq \emptyset$. A finite open cover $\mathcal{U}$ is a tree-cover if the nerve $\mathcal{N}(\mathcal{U})$ is a tree. A finite open cover $\mathcal{U}$ is 1-cyclic if the nerve $\mathcal{N}(\mathcal{U})$ is a graph that contains exactly 1 simple closed curve. A continuum $X$ is tree-like (1-cyclic) if given any $\epsilon > 0$, there is a tree (respectively, 1-cyclic) cover $\mathcal{U}$ of $X$ such that $\text{mesh}(\mathcal{U}) < \epsilon$. Equivalently, $X$ is 1-cyclic if it is the inverse limit of 1-cyclic graphs.

In this section, it will be shown that 1-dimensional 2-separating plane continua are 1-cyclic continua with “degree 1” nested covers. The proof of this is a generalization of the proof due to Bing [1] of the result that all circle-like continua that can be embedded in the plane are the result of the nested intersection of refining circle covers with degree 1.

Let $S$ be a simple closed curve in the plane. Then the interior of $S$ is its bounded complementary domain and the exterior is its unbounded complementary domain. If $Y$ is a subset of the plane, let $\overline{Y}$ denote the complement of $Y$. $W$ is an unbounded plane continuum provided that $W$ is closed, connected and $\overline{W}$ is bounded.

**Theorem 1.** Each 1-dimensional 2-separating continuum $X$ is 1-cyclic. Furthermore, for every $\epsilon > 0$ there exists a 1-cyclic cover of $X$ with mesh less than $\epsilon$ such that each member of that cover is the interior of a disk.

**Proof.** Let $\epsilon > 0$. Let $\mathcal{U}_\epsilon$ be a finite collection of interiors of simple closed curves that covers $X$ with mesh less than $\epsilon$ such that no point is in more than 2 elements of $\mathcal{U}_\epsilon$. Let

$$M = X \cup \{\text{the bounded complementary domain of } X\}$$

and
\[ P = X \cup \{ \text{the unbounded complementary domain of } X \}. \]

Since \( M \) does not separate the plane, there exists an unbounded plane continuum \( W \) which does not intersect \( M \) but with boundary that is a simple closed curve \( S_W \) and is covered by \( U \). Let \( U \) be the collection of all interiors \( \{ U_\alpha \}_{\alpha \in \Gamma} \) of closed curves such that each \( U_\alpha \) is a component of the intersection of \( W \) and an element of \( U \). Let \( U' \) be a minimal finite subcover of \( X \). Notice that no point is in more than 2 elements of \( U' \).

Likewise, there exists a continuum \( Q \) which does not intersect \( P \) such that the boundary is a simple closed curve \( S_Q \) which is covered by \( U' \). Let \( V \) be the collection of all interiors \( \{ V_\beta \}_{\beta \in \Gamma} \) of closed curves such that each \( V_\beta \) is a component of the intersection of \( Q \) and an element of \( U' \). Let \( V' \) be a minimal finite subcover of \( X \). Again, no point is in more than 2 elements of \( V' \). Additionally, each member of \( V' \) is the interior of a disk. Also, notice that \( U' \cup W \cup Q = \mathbb{R}^2 \).

Now to show that \( V' \) is a 1-cyclic cover of \( X \). Suppose \( C_0 = [C_0^0, ..., C_n^0]_0 \) and \( C_1 = [C_1^1, ..., C_m^1]_0 \) are distinct circle-chains of \( V' \). Let \( S_0 = p_1p_2...p_np_1 \) and \( S_1 = r_1r_2...r_mr_1 \) be simple closed curves such that \( p_ip_{i+1}, p_np_1, r_jr_{j+1} \) and \( r_mr_1 \) are arcs contained \( C_0^0, C_0^1, C_1^0, C_1^1 \) respectively. Without loss of generality, we can assume that \( S_Q \) is in the interior of \( S_1 \) and \( S_0 \) is not in the interior of \( S_1 \).

Since \( C_0 \) and \( C_1 \) are distinct, there is an element \( C_0^0 \in C_0 \) that is not in \( C_1 \). So there exists a point \( p \) that is on the boundary of \( C_0^0 \), \( C_0^0 \cap C_1^0 \) and \( C_1^0 \cap C_1^0 \) that is in the interior of \( S_0 \) (which is in the interior of \( S_W \)) but in the exterior of \( S_Q \) ("buffered" by \( S_1 \)). Thus, \( p \notin W \cup Q \) and it follows that \( p \in U \). Hence, there exist distinct \( U_{i-1}, U_i \in U \) such that \( C_0^0 \subset U_i, C_1^0 \subset U_{i-1} \). It now follows from the construction of \( V' \) that \( p \) is on the boundary of \( U_i, U_{i-1}, U_i \cap U_{i-1} \). But then there exists a \( U \subset U \), distinct from \( U_i \) and \( U_{i-1} \), such that \( p \in U \). However, that implies that \( U \cap U_i \cap U_{i-1} = \emptyset \) which contradicts the fact that no point is in more than 2 elements of \( U \). Therefore, \( V \) must have at most 1 cycle.

Next, we will construct a necessary condition for a 1-cyclic continuum to be embeddable in the plane. Let \( \mathcal{C} = [C_0, ..., C_{n-1}]_0 \) be the circle chain of 1-cyclic cover \( \mathcal{U} \). The branch of \( \mathcal{C} \) in \( \mathcal{C} \) is

\[ \mathcal{B}(C_i) = \{ U \in \mathcal{U} \mid \mathcal{D} \text{ is a chain from } U \text{ to } C_i \} \text{ then } \mathcal{D} \cap \mathcal{C} = \{ C_i \}. \]

Let \( \mathcal{U} \) and \( \mathcal{V} \) be 1-cyclic covers such that

1. \( \mathcal{V} \) refines \( \mathcal{U} \).
2. \( \mathcal{C} = [C_0, ..., C_{n-1}]_0 \) is the circle chain cover of \( \mathcal{U} \).
3. \( \mathcal{C}' = [C_0', ..., C_{m-1}]_0 \) is the circle chain cover of \( \mathcal{V} \).
4. Both \( C_0' \) and \( C_{m-1}' \) intersect the core of \( C_0 \).

For \( V \in \mathcal{V} \), define \( \Gamma^\mathcal{V}_\mathcal{U}(V) = j \) if there exists \( U \in \mathcal{B}(C_j) \) such that \( V \subset U \in \mathcal{B}(C_j) \). Note that there will be cases where there are 2 choices for \( \Gamma^\mathcal{V}_\mathcal{U}(V) \). It won’t matter which one is picked until property (7) and Lemma 3. Next define

\[ \Delta^\mathcal{V}_\mathcal{U} : \{0, 1, ..., m-1\} \rightarrow \mathbb{Z}. \]

such that \( \Delta^\mathcal{V}_\mathcal{U}(0) = 0 \) and then continue inductively by
Define the degree of $V$ in $\mathcal{U}$ by

$$\deg_{\mathcal{U}}(V) = \frac{|\Delta^\mathcal{U}_V(m) - \Delta^\mathcal{U}_V(0)|}{n}.$$  

Notice that the degree $V$ in $\mathcal{U}$ is an integer that measures the number of times $C'$ “essentially circles” $C$. Also, since both $C'_0$ and $C'_{m-1}$ intersect the core of $C_0$, this value is independent of our choice for $\Gamma^\mathcal{U}_V(V)$ (when there is a choice).

Next, for each $C'_j \in C'$ define $\Theta^\mathcal{U}_V(C'_j) = 0$. Let $V \in \mathcal{B}(C'_j) - \{C'_j\}$ and let $W$ be the unique element of the chain from $V$ to $C'_j$ in $\mathcal{B}(C'_j)$ that intersects $V$. Then we can define inductively:

$$\Theta^\mathcal{U}_V(V) = \begin{cases}  
\Theta^\mathcal{U}_V(W) & \text{if } \Gamma^\mathcal{U}_V(V) = \Gamma^\mathcal{U}_V(W) \\
\Theta^\mathcal{U}_V(W) + 1 & \text{if } \Gamma^\mathcal{U}_V(V) = \Gamma^\mathcal{U}_V(W) + 1, \\
\Theta^\mathcal{U}_V(W) - 1 & \text{if } \Gamma^\mathcal{U}_V(V) = \Gamma^\mathcal{U}_V(W) - 1, \\
\end{cases}$$

or $\Gamma^\mathcal{U}_V(W) = 0$ and $\Gamma^\mathcal{U}_V(V) = n - 1$.

Finally, for each $V \in \mathcal{V}$, define $\Omega^\mathcal{U}_V : \mathcal{V} \rightarrow \mathbb{Z}$ by

$$\Omega^\mathcal{U}_V(V) = \Delta^\mathcal{U}_V(j) + \Theta^\mathcal{U}_V(V) \text{ where } V \in \mathcal{B}(C'_j).$$

$\Delta^\mathcal{U}_V(j)$ measures the “wrapping position” of $C'_j$ relative to $C'_0$ from the chain $[C'_0, ..., C'_j]$, $\Theta^\mathcal{U}_V(V)$ measures the wrapping position of $V$ relative to $C'_j$ from the chain $[C'_j, ..., W, V]$, and $\Omega^\mathcal{U}_V : \mathcal{V}$ measures the wrapping position of $V$ relative to $C'_0$ from the chain $[C'_0, ..., C'_j, ..., W, V]$. (See Figure 1.)

Let $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$ all be 1-cyclic covers with the following properties:

1. $\mathcal{U}_2$ refines $\mathcal{U}_1$ and $\mathcal{U}_1$ refines $\mathcal{U}_0$.
2. $C'_0 = [C'_0, ..., C'_{n-1}]_0$ is the circle chain cover of $\mathcal{U}_0$.
3. $C'_1 = [C'_0, ..., C'_{m-1}]_0$ is the circle chain cover of $\mathcal{U}_1$.
4. $C'_2 = [C'_0, ..., C'_{m-1}]_0$ is the circle chain cover of $\mathcal{U}_2$.
5. Both $C'_0$ and $C'_{m-1}$ intersect the core of $C'_0$.
6. Both $C'_0$ and $C'_{m-1}$ intersect the cores of both $C'_0$ and $C'_0$.
7. $\Gamma^\mathcal{U}_1(W) = \Gamma^\mathcal{U}_2(W)$ is chosen such that there exists a $V \in \mathcal{U}_1$ such that $W \subset V$ and $\Gamma^\mathcal{U}_1(W) = \Gamma^\mathcal{U}_2(W)$. 
Figure 1. $\Delta_{U_1}^V(91) = 11$, $\Theta_{U_1}^V(V) = 6$, $\Theta_{U_1}^V(W) = -5$, $\Omega_{U_1}^V(V) = 17$ and $\Omega_{U_1}^V(W) = 6$.

**Proposition 2.** For each $C_2^{k-1}, C_2^k \in C_2$, there exist $V, V' \in U_1$, where $C_2^{k-1} \subset V$ and $C_2^k \subset V'$, such that the following are all true for some fixed $\alpha \in \{-1, 0, 1\}$:

1. $\Gamma_{U_2 U_0}^{U_1}(C_2^k) = \Gamma_{U_2 U_0}^{U_1}(C_2^{k-1}) + \alpha \mod n$
2. $\Delta_{U_2 U_0}^{U_1}(k) = \Delta_{U_2 U_0}^{U_1}(k-1) + \alpha$
3. $\Gamma_{U_2 U_0}^{U_1}(V') = \Gamma_{U_2 U_0}^{U_1}(V) + \alpha \mod n$
4. $\Theta_{U_2 U_0}^{U_1}(V') = \Theta_{U_2 U_0}^{U_1}(V) + \alpha$.

**Proof.** Proof follows from property (7) and the constructions of $\Delta_{U_1}^{U_2}$ and $\Delta_{U_0}^{U_2}$.

**Lemma 3.** $\Delta_{U_0}^{U_1}(i) = \Delta_{U_0}^{U_1}(\Delta_{U_1}^{U_2}(i)) + \Theta_{U_0}^{U_1}(V)$ where $C_2^i \subset V$ and $V \in U_1$.

**Proof.** First notice that

$$\begin{align*}
\Delta_{U_0}^{U_1}(0) &= 0 \\
&= \Delta_{U_0}^{U_1}(0) \\
&= \Delta_{U_0}^{U_1}(\Delta_{U_1}^{U_2}(0)) + \Theta_{U_0}^{U_1}(C_0^1).
\end{align*}$$
Then continuing inductively, suppose that $\Delta_{U_0}^{(k)}(k - 1) = \Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(k - 1)) + \Theta_{U_0}^{(k)}(V)$ where $C_{k-1}^2 \subset V$ and $V \in U_1$. Let $V' \in U_1$ such that $C_{k}^2 \subset V'$ as prescribed by property (7). Notice that $V \cap V' \neq \emptyset$.

**Case 1:** $V$ and $V'$ are in the same branch of $U_1$.

Then $\Delta_{U_0}^{(k)}(k) = \Delta_{U_0}^{(k)}(k - 1)$. So it follows from Proposition 2 and the induction hypothesis that

$$
\Delta_{U_0}^{(k)}(k) = \Delta_{U_0}^{(k)}(k - 1) + \alpha
= \Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(k - 1)) + \Theta_{U_0}^{(k)}(V) + \alpha
= \Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(k)) + \Theta_{U_0}^{(k)}(V').
$$

**Case 2:** $V$ and $V'$ are in the different branches of $U_1$.

Then $V, V' \in C_1$ since they intersect. So $V = C_{r_1}^1$ and $V' = C_{r_2}^1$, where $r' = r + \alpha \mod n$ for some $r$. Hence $\Gamma_{U_0}^{(k)}(V') = \Gamma_{U_0}^{(k)}(V) + \alpha \mod n$ and it follows that $\Delta_{U_0}^{(k)}(r') = \Delta_{U_0}^{(k)}(r) + \alpha$. Also, since $C_{k-1}^2 \subset C_{r_1}^1$ and $C_{k}^2 \subset C_{r_2}^1$, we have that $\Delta_{U_0}^{(k)}(k - 1) = r + s\deg_{U_0}(U_1)$ and $\Delta_{U_0}^{(k)}(k) = r' + s\deg_{U_0}(U_1)$. Thus,

$$
\Delta_{U_0}^{(k)}(k) = \Delta_{U_0}^{(k)}(k - 1) + \alpha
= \Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(k - 1)) + \Theta_{U_0}^{(k)}(V) + \alpha
= \Delta_{U_0}^{(k)}(r + s\deg_{U_0}(U_1)) + \Theta_{U_0}^{(k)}(V)
= \Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(k)) + \Theta_{U_0}^{(k)}(V'),
$$

since $\Theta_{U_0}^{(k)}(V) = \Theta_{U_0}^{(k)}(C_{k-1}^2) = 0$ and $\Theta_{U_0}^{(k)}(V') = \Theta_{U_0}^{(k)}(C_{k}^2) = 0$. \hfill $\square$

**Theorem 4.** $\deg_{U_0}(U_2) = \deg_{U_0}(U_1)\deg_{U_1}(U_2)$.

**Proof.** By Lemma 3 it follows that

$$
\deg_{U_0}(U_2) = \frac{\Delta_{U_0}^{(k)}(p) - \Delta_{U_0}^{(k)}(0)}{n}
= \frac{\Delta_{U_0}^{(k)}(\Delta_{U_0}^{(k)}(p)) + \Theta_{U_0}^{(k)}(C_{r_1}^1)}{n}
= \frac{\Delta_{U_0}^{(k)}(n\deg_{U_0}(U_2))}{n}
= \frac{\deg_{U_0}(U_1)\deg_{U_1}(U_2) + \Delta_{U_0}^{(k)}(0)}{n}
= \deg_{U_0}(U_1)\deg_{U_1}(U_2).
$$

\hfill $\square$
**Proposition 5.** Suppose that $\mathcal{V}$ and $\mathcal{U}$ are 1-cyclic covers with the following properties:

1. $\mathcal{V}$ refines $\mathcal{U}$
2. $\mathcal{C}'$ is the circle chain of $\mathcal{V}$ and $\mathcal{C}$ is the circle chain of $\mathcal{U}$
3. There exists a $C_i \in \mathcal{C}$ such that no $C'_j \in \mathcal{C}'$ intersects the core of $C_i$.

then there exists an amalgamation $\mathcal{V}'$ of $\mathcal{V}$ that refines $\mathcal{U}$ such that $\mathcal{V}'$ is a tree cover.

**Proof.** Let $C'_U = \cup \{C'_j \in \mathcal{C}'| C'_j \subset U \text{ for } U \in \mathcal{U} - \{C_i\}\}$. Then $\mathcal{V}' = (\mathcal{V} - \mathcal{C}') \cup \{C'_U| U \in \mathcal{U} - \{C_i\}\}$ is easily verified as a tree cover that refines $\mathcal{U}$. □

**Theorem 6.** If Suppose that $\mathcal{V}$ and $\mathcal{U}$ are 1-cyclic covers such that $\mathcal{V}$ refines $\mathcal{U}$ and $\deg_{\mathcal{U}}(\mathcal{V}) = 0$ then there exists an amalgamation $\mathcal{V}'$ of $\mathcal{V}$ that refines $\mathcal{U}$ such that $\mathcal{V}'$ is a tree cover.

**Proof.** Let $\mathcal{C}'$ be the circle chain of $\mathcal{V}$. Then define $C'_{kU} = \cup \{C'_j \in \mathcal{C}'| C'_j \subset U \text{ and } \Delta_{\mathcal{V}_U}(j) = k \text{ where } U \in \mathcal{U}\}$. Then $\mathcal{V}' = (\mathcal{V} - \mathcal{C}') \cup \{C'_{kU}| U \in \mathcal{U} \text{ and } k \in \mathbb{Z}\}$ is easily verified as a tree cover that refines $\mathcal{U}$.

**Theorem 7.** Suppose that $X$ is an 1-dimensional 2-separating plane continuum and $\epsilon$ is chosen such that every finite open cover of $X$ with mesh less than $\epsilon$ is not a tree cover and $\mathcal{U}$ is a 1-cyclic cover of $X$ with mesh less than $\epsilon$. Then there exists a number $n$ such that if $\mathcal{V}$ is a 1-cyclic cover of $X$ that refines $\mathcal{U}$, then $\deg_{\mathcal{U}}(\mathcal{V}) \leq n$.

**Proof.** Proof is exactly the same a Bing’s proof of Theorem 3 in [1]. Just use the above corresponding results. □

The next corollary is the main result of this section:

**Corollary 8.** If $X$ is an one dimensional 2-separating plane continuum, then $X = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ where $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a sequence of 1-cyclic covers of $X$ with the following properties:

1. $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$
2. $\text{mesh}(\mathcal{U}_i) \to 0$ as $i \to \infty$
3. $\deg_{\mathcal{U}_i}(\mathcal{U}_{i+1}) = 1$.

**Proof.** Follows from Theorems 4, 6 and 7. If more than a finite number refining covers have degree greater than 1, then Theorem 4 will cause a violation of Theorem 7. So just throw out those finite few covers that cause the degree to be greater than 1. □

### 3. Wrapping tree-like subcontinua

Suppose $\mathcal{U}$ is a 1-cyclic cover of $X$, $\mathcal{C} \subset \mathcal{U}$ is the circle-chain of $\mathcal{U}$ and $H$ is a tree-like subcontinuum of $X$ such that $H \cap C \neq \emptyset$ for some $C \in \mathcal{C}$. Define $T(H,\mathcal{U})$ to be a tree-cover of $H$ that refines $\mathcal{U}$ and has minimum cardinality. Define $Q(H,\mathcal{U}) = \{Q \in T(H,\mathcal{U})| Q \subset C \text{ for some } C \in \mathcal{C}\}$ as the trunk of $T(H,\mathcal{U})$.

**Proposition 9.** $Q(H,\mathcal{U})$ is a connected collection.
Suppose that \( Q(H, U) \) is not connected, then there exist nonempty disjoint sets \( Q_a, Q_b \) such that \( Q(H, U) = Q_a \cup Q_b \) and for every \( Q_A \in Q_a \) and \( Q_B \in Q_b \) there is no chain in \( Q(H, U) \) from \( Q_A \) to \( Q_B \). Let \( Q^a \in Q_a \) and \( Q^b \in Q_b \). Since \( T(H, U) \) is a tree-cover there exists a unique chain \( \mathcal{C} = [T_0, T_1, ..., T_n] \subset T(H, U) \) where \( T_0 = Q^a \) and \( T_n = Q^b \).

Let \( j_a \) be the largest integer in \([0, ..., n - 1]\) such that \( T_{j_a} \in Q_a \) and let \( j_b \) be the smallest integer of \([1, ..., n]\) greater than \( j_a \) such that \( T_{j_b} \in Q_b \). If \( j_a + 1 = j_b \), then \([T_{j_a}, T_{j_b}] \) is a chain from \( Q_a \) to \( Q_b \), contained in \( Q(H, U) \), which is a contradiction. So suppose that \( j_b > j_a + 1 \).

Then there exists \( U_{j_a+1} \in U - C \) such that \( T_{j_a+1} \subset U_{j_a+1} \). Let \( C_{j_a}, C_{j_b} \in C \) such that \( T_{j_a} \subset C_{j_a}, T_{j_b} \subset C_{j_b} \), and define \( U_{ab} = \{ U \in U | T_i \subset U \text{ for } i \in \{j_a + 1, ..., j_b - 1\} \} \). Since \([T_{j_a+1}, \ldots, T_{j_b-1}] \) is connected, \( U_{ab} \) is connected. Also note that each \( T_i \in U \setminus Q_b \) and hence, \( T_i \subset U \setminus C \) for each \( i \in \{j_a + 1, ..., j_b - 1\} \). Thus, \( U_{ab} \cap C = \emptyset \). It follows that \( U_{ab} \) contains no circle-chain and hence must be a tree-cover. Notice that since \( T_{j_a+1} \subset U_{j_a+1} \) it follows that \( U_{j_a+1} \in U_{ab} \) and since \( T_{j_a+1} \cap T_{j_b} \neq \emptyset \), we may conclude that \( U_{j_a+1} \cap C_{j_b} \neq \emptyset \). Thus, \( U_{ab} \subset B(C_{j_b}) \). Likewise, \( U_{j_b-1} \setminus C_{j_b} \neq \emptyset \) and thus, \( U_{ab} \subset B(C_{j_b}) \). So \( B(C_{j_a}) \cap B(C_{j_b}) \neq \emptyset \), and it follows that \( C_{j_a} = C_{j_b} \). Therefore, \( j_a = j_b \) which is a contradiction.

**Proposition 10.** Suppose that \( A, B, \) and \( C \) are distinct elements of \( T(H, U) \) such that \( A \cap B \neq \emptyset \) and \( C \cap B \neq \emptyset \). If \( U, V \in U \) such that \( A \subseteq U \) and \( C \subseteq V \), then \( U \neq V \).

**Proof.** If \( U = V \) then \( (T(H, U) - \{A, C\}) \cup \{A \cup C\} \) would be a tree-cover of \( H \) that refines \( U \) but has cardinality less than \( T(H, U) \), which is a contradiction.

**Lemma 11.** Let and \( C = [C_0, ..., C_{n-1}] \) be the circle-chain of \( U \). Then \( Q(H, U) \) is a chain of the form \([C_0(0), C_1(0), ..., C_{n-1}(0), C_0(1), ..., C_{n-1}(1), C_0(2), ..., C_1(j)]\) where \( C_k(m) \subseteq C_k \) for each \( m \).

**Proof.** Since the nerve of \( Q(H, U) \) is a tree, it must have an endlink, say \( C_0(0) \). Let \( C_1(0) \) be the unique link of \( Q(H, U) \) that intersects \( C_0(0) \). Next, rename the elements of \( C \), now called \( C^H \), in the following way: \( C^H = [C_0, C_2, ..., C_{n-1}]^H \) where \( C_0(0) \subseteq C_0 \) and \( C_1(0) \subseteq C_1 \).

Let \( Q \) be an element of \( Q(H, U) \) different from \( C_0(0) \) that intersects \( C_1(0) \). It follows from Proposition 10 that \( Q \subset C_2 \). Let \( C_2(0) = Q \). It also follows from Proposition 10 that no other element of \( Q(H, U) \) can intersect \( C_1(0) \).

Continuing inductively, suppose that \([C_0(0), ..., C_k(m)]\) have been found and \( k < n - 1 \). Let \( Q' \) be an element of \( Q(H, U) \) different from \( C_{k+1}(m) \) that intersects \( C_k(m) \). It follows again from Proposition 10 that \( Q' \subset C_{k+1} \). Let \( C_{k+1}(m) = Q' \). Again, it also follows from Proposition 10 that no other element of \( Q(H, U) \) can intersect \( C_k(m) \). If \( k = n - 1 \), then in a similar way we can let \( C_0(m+1) = Q' \). Since \( Q(H, U) \) is finite, this process will stop at endlink \( C_1(j) \).

Let

\[
\mathcal{B}(C_k(m)) = \{ V \in T(H, U) | \text{there exists a chain } \mathcal{L} \subseteq T(H, U) \text{ from } C_k(m) \text{ to } V \text{ such that } \mathcal{L} \cap Q(H, U) = \{C_k(m)\}. \}
\]

We can think of \( \mathcal{B}(C_k(m)) \) as a “branch” of \( T(H, U) \) connected to the trunk \( Q(H, U) \) at \( C_k(m) \). If \( B \in \mathcal{B}(C_k(m)) \) and \( U \in U \) is the unique element such that \( B \subset U \), then change the name of \( B \) to \( U(m) \). Hence, in the nomenclature of \( T(H, U), \)
Suppose that $T$.

It follows from Proposition 12 that if $U$ circle-chain in $U = U$ the elements have the same symbol, then they must differ in index. Then the new element of $U$ does not have the same symbol, then their union cannot be contained in any closure 2-refinement of $U$.

Let $T$.

Next let $U_0$ and $U_1$ be 1-cyclic covers of $X$ with such that $U_1$ is a degree 1 closure 2-refinement of $U_0$. Let $C_0 = [C^0_0, ..., C^0_{n−1}]_0$ and $C_1 = [C^1_0, ..., C^1_{m−1}]_0$ be the respective circle chains of $U_0$ and $U_1$. Define the wrapping number $W(U_1, U_0)$ of $U_1$ on $U_0$ as

$$W(U_1, U_0) = \max \{ i | \Omega^{i}_{U_0}(V) | V \in U_1 \} - \min \{ \Omega^{i}_{U_0}(V) | V \in U_1 \}.$$ 

$W(U_1, U_0)$ counts the maximum number of times that any subchain of $U_1$ wraps $U_0$. Then, for each $U \in U_0$, let

$$\hat{U}(i + j) = \bigcup \{ V(j) \in T(H, U_1) | V \subset U \text{ and } i = [\frac{\Omega^{i}_{U_0}(V)}{n}] \}.$$ 

Let $T$ be the collection of all such $\hat{U}(i + j)$.

**Proposition 12.** Suppose that $V, V' \in U_1$ such that $V \cap V' \neq \emptyset$. Then

$$|\Omega^{i}_{U_0}(V) - \Omega^{i}_{U_0}(V')| = 0, 1 \text{ or } n.$$ 

**Proof.** Follow from the fact that $\deg_{U_0}(U_1) = 1$. □

**Lemma 13.** $T$ is tree-cover of $H$ that refines $U_0$ that has minimum cardinality.

**Proof.** It follows from Proposition 12 that if $\hat{U}(k), \hat{V}(m) \in T$ then $\hat{U}(k) \cap \hat{V}(m) \neq \emptyset$ if and only if $i = m$ and $\hat{U}, \hat{V}$ are elements of $U_0$ that intersect or if $k = m - 1, \hat{U} = C^0_{n−1}$ and $\hat{V} = C^0_0$ (or vice versa). Hence, a circle-chain in $T$ would imply a circle-chain in $U_0$ composed of the same symbols. But the only circle-chain in $U_0$ is $[C^0_0, C^0_1, ..., C^0_{n−1}]_0$. This would imply that the circle-chain of $T$ must be of the form $[\hat{C}^0_0(i), \hat{C}^0_1(i), ..., \hat{C}^0_{n−1}(i)]_0$ for some $i$. However that implies that $\hat{C}^0_0(i)$ and $\hat{C}^0_{n−1}(i)$ intersect which is impossible. Hence, $T$ is a tree-cover of $H$.

To show that $T$ is minimal, union any 2 distinct elements of $T$ together. If they do not have the same symbol, then their union cannot be contained in any element of $U_0$ and the new cover is no longer a refinement. On the other hand, if the elements have the same symbol, then they must differ in index. Then the new cover will contain a circle-chain and thus, will no longer be a tree-cover. □

Let $i_{\min} = \min \{ i | \hat{U}(i) \in T \}$. Then it follows that

$$\min \{ [\frac{\Omega^{i}_{U_0}(V)}{n}] | V \in U_1 \} \leq i_{\min} \leq 0.$$ 

Let $T(H, U_0) = T$ with the elements renamed in the following way: $U(j - i_{\min}) = \hat{U}(j)$. Then the trunk $\Omega(H, U_0)$ of $T(H, U_0)$ is in the form of Lemma 11.

**Proposition 14.** $W(H, U_0) \leq W(H, U_1) + W(U_1, U_0)$. 

**Proof.** Follow from the fact that $\deg_{U_0}(U_1) = 1$. □
Suppose that $W(U, U_0) = \max \{i | U(i) \in T(H, U_0) \text{ and } U \in U_0\}$

\[ \leq \max \{j | V(j) \in T(H, U_1) \text{ and } V \in U_1\} + \max \left\{ \left\lfloor \frac{\Omega_{U_0}^U(V)}{n} \right\rfloor | V \in U_1 \right\} - i_{\min} \]

\[ \leq W(H, U_1) + W(U_1, U_0). \]

\[ \square \]

**Proposition 15.** Suppose that $H$ is a tree-like subcontinuum of $X$ and $U_0, U_1$ are finite 1-cyclic open covers of $X$ such that $U_1$ is a degree 1, 2-refinement of $U_0$. Then $W(H, U_1) \leq W(H, U_0)$.

**Proof.** Let $p = W(H, U_1)$. Then there exists $C_1^k \in C_1$ such that

\[ C_1^k(0), C_1^k(1), ..., C_1^k(p) \in T(H, U_1). \]

Let $i = \left\lfloor \frac{\Omega_{U_0}^U(C_1^k)}{n} \right\rfloor$ and $U \in U_0$ such that $C_1^k \subset U$. Then

\[ \bar{U}(i + 0), \bar{U}(i + 1), ..., \bar{U}(i + p) \in T. \]

Hence $p \leq W(H, U_0)$. \[ \square \]

**Theorem 16.** Let $h : X \rightarrow X$ be a homeomorphism. Suppose that $H$ is a tree-like subcontinuum of $X$ and $U_0$ and $U_1$ are finite 1-cyclic open covers of $X$ such that both $U_1$ and $h(U_1)$ are degree 1, 2-refinements of $U_0$. Then

\[ W(h^n(H), U_0) \leq W(H, U_0) + nW(h(U_1), U_0). \]

**Proof.** Proof is by induction on $n$.

**Base Case.** If $n = 0$ then the theorem is clearly true.

**Induction step.** Suppose that $W(h^{n-1}(H), U_0) \leq W(H, U_0) + (n-1)W(h(U_1), U_0)$.

By Proposition 15, $W(h^{n-1}(H), U_1) \leq W(h^{n-1}(H), U_0)$. Since $W(h^n(H), h(U_1)) = W(h^{n-1}(H), U_1)$, it follows from the induction hypothesis and Proposition 14 that

\[ W(h^n(H), U_0) \leq W(h^n(H), h(U_1)) + W(h(U_1), U_0) \leq W(H, U_0) + nW(h(U_1), U_0). \]

\[ \square \]

4. Main result

Entropy is a measure of how fast points move apart and expansive homeomorphisms have positive entropy. The following definition of entropy is due to Bowen [7]: If $h : X \rightarrow X$ is a map and $n$ a non-negative integer, define

\[ d^+_n(x, y) = \max_{0 \leq i < n} d(h^i(x), h^i(y)). \]

Similarly, if $h$ is a homeomorphism, define

\[ d^-_n(x, y) = \max_{-n < i \leq 0} d(h^i(x), h^i(y)), \]

where again $n \geq 0$. 

Let $K$ be a compact subset of $X$ and $n$ be a positive integer. A finite subset $E_n$ of $K$ is said to be $(n, \epsilon)$-separated with respect to map $h$ if $x$ and $y$ are distinct elements of $E_n$ implies that $d^+_n(x, y) > \epsilon$. Let $s_n(\epsilon, K, h)$ denote the largest cardinality of any $(n, \epsilon)$-separated subset of $K$ with respect to $h$. Then

$$s(\epsilon, K, h) = \limsup_{n \to \infty} \frac{\log s_n(\epsilon, K, h)}{n}.$$ 

The entropy of $h$ on $X$ is then defined as

$$\text{Ent}(h, X) = \sup_{\epsilon > 0} \lim_{\epsilon \to 0} s(\epsilon, K, h) | K \text{ is a compact subset of } X \}.$$ 

A subcontinuum $M$ of $X$ is stable under homeomorphism $h : X \to X$ if $\lim_{n \to \infty} \text{diam}(h^n(M)) = 0$. Likewise, $M$ is unstable under $h$ if $\lim_{n \to \infty} \text{diam}(h^n(M)) = 0$.

The following theorems are due to Kato. The second is found in the proof of Theorem 4.1 of [3].

**Theorem 17.** If $h : X \to X$ is an expansive homeomorphism, then there exists a stable subcontinuum or an unstable subcontinuum.

**Theorem 18.** If $h : X \to X$ is an expansive homeomorphism and $M$ is an unstable subcontinuum, then there exists an $\epsilon > 0$ such that $s(\epsilon, M, h) > 0$.

Likewise, if $M$ is a stable subcontinuum, then there exists an $\epsilon > 0$ such that $s(\epsilon, M, h^{-1}) > 0$.

The proof of the main result now follows in a similar way to the proof that tree-like continua do not admit expansive homeomorphisms [4]. The next proposition is Cantor’s original definition of connected [2].

**Proposition 19.** Suppose $X$ is connected and $a, b \in X$. For every $\epsilon > 0$ there exists a finite sequence $\{x_i\}_{i=1}^n \subset X$ such that $x_1 = a$, $x_n = b$ and $d(x_i, x_{i+1}) < \epsilon$.

The previous sequence is called a simple chain sequence from $a$ to $b$ with mesh less than $\epsilon$. If $h$ is a homeomorphism and $n$ a positive integer, define $L(h, n, \epsilon)$ to be a number greater than $0$ such that

$$d(x,y) < L(h, n, \epsilon) \implies d(h^i(x), h^i(y)) < \epsilon$$

for all $-n \leq i \leq n$.

**Lemma 20** (4). Suppose that $h : X \to X$ is a homeomorphism of a continuum $X$ and that $\{x_i\}_{i=1}^m$ is a simple chain sequence of $X$ from $a$ to $b$ with mesh less than $L(h, n, \epsilon/6)$. Also, suppose that $\{x_i\}_{i=1}^m$ is contained in some tree-cover $T$ such that $a$ and $b$ are in the same element $T_1$ of $T$ and that the mesh of $\{x_i\}_{i=1}^m$ is less than the Lebesgue number of $T$. If $d_n(a, b) \geq \epsilon$, then there exist $x_\alpha, x_\beta \in \{x_i\}_{i=1}^m$ such that $x_\alpha, x_\beta$ are in the same element of $T$ and $\epsilon/3 \leq d_n(a, x_\beta) < \epsilon$.

**Lemma 21** (4). Let $h : X \to X$ be a homeomorphism of a compact space onto itself. Suppose that there are sequences $\{y_i\}_{i=1}^\infty, \{z_i\}_{i=1}^\infty$ such that $d(h^k(y), h^k(z)) < \epsilon$ for all $k \leq n$. Then there exists a limit point $y$ of $\{y_i\}_{i=1}^\infty$ and a limit point $z$ of $\{z_i\}_{i=1}^\infty$ such that $d(h^k(y), h^k(z)) < 2\epsilon$ for all $k$.

**Theorem 22.** Degree 1, 1-cyclic continua do not admit expansive homeomorphisms.
Suppose that 1-dimensional plane continua that have 2 complementary domains. Does there exist a 1-dimensional 3-separating plane continuum that admits an expansive homeomorphism? The pseudo-circle does not admit an expansive homeomorphism.

Question 1.

Corollary 23.

Corollary 24. The pseudo-circle does not admit an expansive homeomorphism.

Proof. Suppose that \( h : X \to X \) is an expansive homeomorphism of a 1-cyclic plane continuum with expansive constant \( c \). By Theorem 17, there exists a nondegenerate stable or unstable subcontinuum \( M \). Without loss of generality, we may assume that \( \text{diam}(h^i(M)) < c/3 \) for all \( i \). It follows that \( M \) must be tree-like. By Theorem 18, there exists a \( \gamma > 0 \) such that \( s(\gamma, M, h) > 0 \). Let \( \epsilon = \min\{\gamma, c/4\} \) and \( \{\delta_k\}_{k=1}^\infty \) be a sequence of positive numbers such that each \( \delta_k < \epsilon \). Let \( \{U_k\}_{k=0}^\infty \) be a sequence of 1-cycle covers of \( X \) such that

1) \( \text{mesh}(U_k) < \delta_k \)
2) Both \( U_{k+1} \) and \( h(U_{k+1}) \) are 1-degree, closure 2-refinements of \( U_k \).

By Theorem 16,

\[
|T(h^n(M), U_k)| \leq |U_k|W(h^n(M), U_k) \leq |U_k|(W(M, U_k) + n\tilde{W}(h(U_{k+1}), U_k))
\]

which has polynomial growth as \( n \) increases. Since \( s(\epsilon, M, h) > 0 \), \( s_n(\epsilon, M, h) \) must have exponential growth as \( n \) increases. Therefore, for some integer \( N_k \),

\[
s_{N_k}(\epsilon, M, h) > |T(h^{N_k}(M), U_k)| + 1.
\]

Let \( E_k^N \) be the maximal \((N_k, \epsilon)\)-separated set of \( M \). Then by the pigeon-hole principle, there exists \( a^k, b^k \in E_k^N \) such that \( h^{N_k}(a^k), h^{N_k}(b^k) \) are in a common element of \( T(h^{N_k}(M), U_k) \). Since \( d_{N_k}(h^{N_k}(a^k), h^{N_k}(b^k)) \geq \epsilon \), it follows from Lemma 20 that there exists \( x_{\alpha}^k, x_{\beta}^k \in h^{N_k}(M) \) such that \( \epsilon/3 \leq d_{N_k}(x_{\alpha}^k, x_{\beta}^k) < \epsilon \) and \( d(x_{\alpha}^k, x_{\beta}^k) < \delta_k \). Hence, \( d(h^i(x_{\alpha}^k), h^i(x_{\beta}^k)) < \epsilon \) for all \( i \leq k \).

Now let \( m_k \in \{0, \ldots, N_k - 1\} \) such that \( d(h^{m_k}(x_{\alpha}^k), h^{m_k}(x_{\beta}^k)) \geq \epsilon/3 \). For ease of notation, define \( y_k = h^{m_k}(x_{\alpha}^k) \) and \( z_k = h^{m_k}(x_{\beta}^k) \). Then \( d(h(y_k), h(z_k)) < \epsilon \) for all \( i < k + m_k \). By Lemma 21, there exist limit points \( y \) of \( \{y_k\}_{k=1}^\infty \) and \( z \) of \( \{z_k\}_{k=1}^\infty \) such that \( d(h^i(y), h^i(z)) \leq 2\epsilon < \epsilon \) for all \( i \). However, since \( d(y_k, z_k) \geq \epsilon/3 \) for all \( k > 0 \), \( y \) and \( z \) must be distinct. Therefore, \( h \) is not expansive.

The following interesting results now follow:

Corollary 23. 1-dimensional plane continua that have 2 complementary domains do not admit expansive homeomorphisms.

Corollary 24. The pseudo-circle does not admit an expansive homeomorphism.

The following questions remain open.

Question 1. Does there exist a 1-dimensional 3-separating plane continuum that admits an expansive homeomorphism?

Question 2. Does there exist a 2-dimensional non-separating plane continuum that admits an expansive homeomorphism?

5. References


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