CONTINUA THAT ADMIT EXPANSIVE $\mathbb{Z}^{n+1}$ ACTIONS BUT NOT EXPANSIVE $\mathbb{Z}^n$ ACTIONS

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Abstract. A collection $\{h_1, \ldots, h_n\}$ of commutative homeomorphisms on $X$ generate an expansive $\mathbb{Z}^n$ action on $X$ provided that for some fixed $c > 0$ and every distinct $x, y \in X$ there exist integers $k_1, \ldots, k_n$ such that
$$d(h_1^{k_1} \circ h_2^{k_2} \circ \ldots h_n^{k_n}(x), h_1^{k_1} \circ h_2^{k_2} \circ \ldots h_n^{k_n}(y)) \geq c.$$ 

For each set of positive integers $k$ and $n$, a $k$-dimensional continuum $X_k^n$ is constructed such that $X_k^n$ admits an expansive $\mathbb{Z}^{n+1}$ action but does not admit an expansive $\mathbb{Z}^n$ action.

1. Introduction

A homeomorphism $h : X \to X$ is expansive provided that for some fixed $c > 0$ and every distinct $x, y \in X$ there exists an integer $n$, dependent only on $x$ and $y$, such that
$$d(h^n(x), h^n(y)) \geq c.$$ 

Here $c$ is called the expansive constant. The dyadic solenoid and the torus are examples of continua that do admit expansive homeomorphisms. The arc and the simple closed curve are continua that do not admit expansive homeomorphisms.

A collection $\{h_1, \ldots, h_n\}$ of commutative homeomorphisms on $X$ generate an expansive $\mathbb{Z}^n$ action on $X$ provided that for some fixed $c > 0$ and every distinct $x, y \in X$ there exist integers $k_1, \ldots, k_n$ such that
$$d(h_1^{k_1} \circ h_2^{k_2} \circ \ldots h_n^{k_n}(x), h_1^{k_1} \circ h_2^{k_2} \circ \ldots h_n^{k_n}(y)) \geq c.$$ 

Clearly, if $n = 1$, then we have an expansive homeomorphism. A major question on this topic is “What continua admit expansive $\mathbb{Z}^n$ actions and what continua do not?”. Shi, Zhou and Zhou have shown the nonexistence of expansive $\mathbb{Z}^n$ actions on Peano continua that contain no $\theta$-curves [8]. However, there is currently not much known on the topology of continua that admit expansive $\mathbb{Z}^n$ actions. The purpose of this paper is to show that for any natural numbers $n$ and $m$, there exists an $m$-dimensional continuum that admits an expansive $\mathbb{Z}^{n+1}$ action but does not admit an expansive $\mathbb{Z}^n$ action. Mané proved that there is no expansive homeomorphism on infinite dimensional continua [3]. However, Shi and Zhou constructed an infinite dimensional continuum that admits an expansive $\mathbb{Z}^2$ action [9]. In the last section, other questions will be asked and examples constructed.

2. Continua that admit expansive $\mathbb{Z}^{n+1}$ actions but not expansive $\mathbb{Z}^n$ actions

Let $X$ be a topological space and $f : X \to X$ be a continuous function. The inverse limit of $(X, f)$ is a new topological space:
$$\tilde{X} = \varprojlim_{i=1}^{\infty} \{X, f\}_{i} = \{\langle x_i \rangle_{i=1}^{\infty} \mid x_i \in X \text{ and } f(x_{i+1}) = x_i\}.$$ 

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$\hat{X}$ has the subspace topology induced on it by $\prod_{i=1}^{\infty} X$. If $\langle x_i \rangle_{i=1}^{\infty}, \langle y_i \rangle_{i=1}^{\infty} \in \hat{X}$ then define the metric on $\hat{X}$ by

$$\hat{d}(\langle x_i \rangle_{i=1}^{\infty}, \langle y_i \rangle_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}$$

where $d$ is the metric on $X$. If $f$ is a homeomorphism on $X$, then it can easily be shown that $\hat{X}$ is homeomorphic to $X$. Furthermore, the shift homeomorphism $\hat{f}$ on $\hat{X}$ is defined as

$$\hat{f}(\langle x_i \rangle_{i=1}^{\infty}) = (\langle f(x_i) \rangle_{i=1}^{\infty}) = (\langle f(x_1), x_1, x_2, ... \rangle).$$

Notice that $\hat{f}^{-k+1}(\langle x_i \rangle_{i=1}^{\infty}) = \langle x_i \rangle_{i=k}^{\infty}$.

Suppose that $\hat{X} = \lim_{i \to \infty} (X, f)^{\infty}$. Then the shift map $\hat{f}$ is an expansive homeomorphism on $\hat{X}$ with expansive constant $c$.

1. If $\hat{X} = \lim_{i \to \infty} (X, f)^{\infty}$ then the shift map $\hat{f}$ is an expansive homeomorphism on $\hat{X}$ with expansive constant $c$.

2. There exists an arc $I(a, f(a))$ in $X$ from $a$ to $f(a)$ and an arc $I(f(a), f^2(a))$ such that for every distinct $x, y \in I(a, f(a)) \cup I(f(a), f^2(a))$, there exists a non-negative integer $n$ such that $d(f^n(x), f^n(y)) > c$.

Then we say that $(X, f)$ has Property $P$.

Theorem 1. Suppose that $(X, f)$ has Property $P$. Then there exists an extension $F$ of $f$ on $X(a, n)$ such that the shift map $\hat{F}$ on $\hat{X}(a, n) = \lim_{i \to \infty} \{X(a, n), F\}^{\infty}_{i=1}$ is expansive.

Proof. Let $A_i$ be the arc $[0, 1]_i \cup I[a, f(a)]$ where $1_i$ is identified to $a$. Let $L_i : [0, 1]_i \to A_i$ be a function that takes $[0, 1/2]_i$, linearly onto $[0, 1]_i$ and $[1/2, 1]_i$ homeomorphically onto $I[a, f(a)]$. Then define

$$F(x) = \begin{cases} f(x) & \text{if } x \in X \\ L_i(x) & \text{if } x \in [0, 1]_i \end{cases}$$

Let $R_i = \{\langle x_j \rangle_{j=0}^{\infty} \in \hat{X}(a, n) | x_j \in [0, 1]_i \text{ for some } j \}$. Then each $R_i = [0, \infty)$ is a ray in $\hat{X}(a, n)$ that limits to a subset of $\hat{X}$ and $\hat{X}(a, n) = \hat{X} \cup \bigcup_{i=1}^{n} R_i$. We will now show that $\hat{F}$ is expansive on $\hat{X}(a, n)$.

Let $c_1 = \min\{d(X, [0, 1/2]_i) | i \in \{1, ..., n\}\}$

and

$c_2 = \min\{d([0, 1/2]_i, [0, 1/2]_j) | i, j \text{ are distinct elements of } \{1, ..., n\}\}.$

The expansive constant for $\hat{F}$ will be $c' = \min\{c, c_1, c_2\}$. Suppose that $x = \langle x_j \rangle_{j=0}^{\infty}$ and $y = \langle y_j \rangle_{j=0}^{\infty}$ are distinct elements in $\hat{X}(a, n)$. There exists a $m$ such that $x_m \neq y_m$. We have 4 cases to consider:

Case 1: $x, y \in \hat{X}$.

Then by the hypothesis of the theorem there exists a $k$ such that

$$\hat{d}(\hat{F}^k(x), \hat{F}^k(y)) = d(f^k(x), f^k(y)) > c \geq c'.$$

Case 2: $x, y \in R_i$ for some $i \in \{1, ..., n\}$. 


There exists an integer $k$ such that $x_k, y_k \in [0, 1/2]$, and $x_k \neq y_k$. Without loss of generality assume $x_k > y_k$. Then there exists a smallest positive integer $p$ such that $L^p(x_k) \in I(a, f(a))$. There are 3 subcases to consider:

**Case 2a:** $L^p(y_k) \in [0, 1/2]$.  
Then $d(L^p(x_k), L^p(y_k)) \geq c$. Hence, $\hat{d}(\hat{F}^{p-k+1}(x), \hat{F}^{p-k+1}(y)) \geq c'$.

**Case 2b:** $L^p(y_k) \in [1/2, 1]$.  
Then $L^{p+1}(y_k) \in I(a, f(a))$ and $f(L^{p+1}(x_k)) \in I(f(a), f^2(a))$. So there exists a nonnegative integer $q$ such that $d(f^q(L^{p+1}(x_k)), f^q(L^{p+1}(y_k))) \geq c$. Hence, $\hat{d}(\hat{F}^{q+p-k+2}(x), \hat{F}^{q+p-k+2}(y)) \geq c'$.

**Case 2c:** $L^p(y_k) \in I(a, f(a))$.  
Then there exists a nonnegative integer $q$ such that $d(f^q(L^p(x_k)), f^q(L^p(y_k))) \geq c$. Hence, $\hat{d}(\hat{F}^{q+p-k+1}(x), \hat{F}^{q+p-k+1}(y)) \geq c'$.

**Case 3:** $x \in R_i$ and $y \in R_j$ for distinct $i, j \in \{1, ..., n\}$.  
Then there exists an integer $k$ such that $x_k \in [0, 1/2]_i$ and $y_k \in [0, 1/2]_j$. So $d(x_k, y_k) \geq c_2$. Hence $\hat{d}(\hat{F}^{k+1}(x), \hat{F}^{k+1}(y)) \geq c'$.

**Case 4:** $x \in R_i$ and $y \in \bar{X}$. Then there exists an integer $k$ such that $x_k \in [0, 1/2]_i$. Since $y_k \in X$ it follows that $d(x_k, y_k) \geq c_1$. Hence $\hat{d}(\hat{F}^{k+1}(x), \hat{F}^{k+1}(y)) \geq c'$.

Let $S$ be the unit circle $[0, 1]_0$ where 0 is identified with 1 and let $\sigma : S \longrightarrow S$ by $\sigma(z) = 2z \mod 1$. Define

$$\Sigma_2 = \lim_{\circlearrowleft i=1} S, \sigma_i^\infty.$$  
Then $\Sigma_2$ is the dyadic solenoid. It is well-known that the shift homeomorphism is expansive with expansive constant $c = 1/8$.

**Theorem 2.** $(S, \sigma)$ has property $P$.

**Proof.** Let $a = 1/32$, then $\sigma(a) = 1/16$ and $\sigma^2(a) = 1/8$. Pick distinct $x, y \in [1/32, 1/8]$. Then there exists a positive integer $n$ such that $1/8 < 2^n|y-x| \leq 1/4$. Hence $d(\sigma^n(x), \sigma^n(y)) > c$. \hfill $\square$

Let $\sigma_n : S(1/32, n) \longrightarrow S(1/32, n)$ be the extension of $\sigma$. Then by Theorem 1 the shift map $\sigma_n$ on $\Sigma_2 = \lim_{\circlearrowleft i=1} S(a, n), \sigma_i^\infty$ is expansive. Next, let $T_k = \prod_{i=1}^k S = \mathbb{R}^k / \mathbb{Z}$ be the $k$-torus.

**Theorem 3 (7).** $T_k$ admits an expansive homeomorphism for $k \geq 2$.

Let

$$A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$  

Then define $A_2 = A$, $A_3 = B$ and for $k \geq 2$

$$A_{2k} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \text{ and } A_{2k+1} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{bmatrix}.$$  

It has been shown in [7] that $A_k$ is an expansive homeomorphism of $T_k$ for $k \geq 2$.

**Theorem 4.** $T_k$ has property $P$ for $k \geq 2$.  

Proof. Proof will be for $T_{2k}$. Proof for $T_{2k+1}$ is similar. For ease of notation let $f = A_{2k}$. Since $f$ is an expansive homeomorphism, the shift map induced by $f$ is also expansive.

For condition 2, let $a = [1/16,0,...,0]^T$. Then $f(a) = [1/16,1/16,0,...,0]^T$ and $f^2(a) = [1/8,3/16,0,...,0]^T$. Let $x = [x_1,x_2,0,...,0]^T$ and $y = [y_1,y_2,0,...,0]^T$. So if $x$, $y \in I(a, f(a)) \cup I(f(a), f^2(a))$ then notice that $x_1 \leq y_1$ if and only if $x_2 \leq y_2$. Let $p = x - y$ and have the expansive constant be $1/4$. We want to find an integer $q$ such that

$$||A_{2k}^q p|| = ||A_{2k}^q (x - y)|| = ||A_{2k}^q x - A_{2k}^q y|| \geq 1/4.$$ 

First notice that if $j$ is a positive integer, then

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \\ \vdots & \vdots & \vdots \end{bmatrix}^{j} = \begin{bmatrix} F_{2j-1} & F_{2j} \\ F_{2j} & F_{2j+1} \end{bmatrix}$$

and that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \\ \vdots & \vdots & \vdots \end{bmatrix}^{-j} = \begin{bmatrix} F_{2j+1} & -F_{2j} \\ -F_{2j} & F_{2j+1} \end{bmatrix}$$

where $F_n$ is the $n$th Fibonacci number. Also notice that if $||p|| < 1/4$ then $||Ap|| < 1$. Let $p = [p_1,p_2,0,...,0]^T$. Since $x$ and $y$ are assumed distinct, $p_1$ and $p_2$ are not both 0. Also, since $x, y \in I(a, f(a)) \cup I(f(a), f^2(a))$, we have either $0 \leq p_1 < 1/4$ and $0 \leq p_2 < 1/4$ or $0 \leq p_1 < 1/4$ and $0 \leq -p_2 < 1/4$. Then $A_{2k}^q p = [F_{2j-k} p_1 + F_{2j} p_2, F_{2j+1} p_1 + F_{2j+2} p_2, 0,...,0]^T$. Hence $||A_{2k}^q p|| \geq \max\{F_{2j-k}, F_{2j+1}\}$ Since $F_{2j}$ can be made arbitrary large, there exists a positive integer $q$ such that $||A_{2k}^q x - A_{2k}^q y|| \geq 1/4$. \hfill \Box

Let $W(1,n) = \Sigma^\infty_{-\infty}$ and for $k > 1$ let $W(k,n) = \lim\{T_k(a,2n), G\}_{n=1}^\infty$ where $G$ is the extension of $A_k$ to $T_k(a,2n)$ as in Theorem 1. Consider the set $\{W_\alpha(k,n) | \alpha \in \mathbb{Z}^n\}$. Then $W_\alpha(k,n) = \tilde{X}_\alpha \cup \bigcup_{i=1}^{2n} R_\alpha^i$ where $\tilde{X}_\alpha$ is either the solenoid or the $k$-torus and each $R_\alpha^i$ is a ray that limits to a subset of $\tilde{X}_\alpha$. Let $\pi_i$ be the $i$th coordinate map of $\alpha \in \mathbb{Z}^n$. Suppose that for some $i \in \{1,2,...,n\}$ we have $\pi_i(\alpha) = \pi_i(\alpha') - 1$ and $\pi_j(\alpha) = \pi_j(\alpha')$ for $j \neq i$. Then identify the endpoint $0_{2n}^\alpha$ of $R_\alpha^0$ to the endpoint $0_{2n}^{\alpha'}$ of $R_\alpha^{\alpha'}$ to $W_\alpha(k,n)$. Let $Z(k,n)$ be the resulting space. For $w \in W(k,n)$ we say that $(w, \alpha)$ is the corresponding point of $W_\alpha(k,n)$ in $Z(k,n)$. The compactification of $Z(k,n)$ by identifying $\infty$ with a point, denoted by $Z(k,n)/\infty$, is a $k$-dimensional continuum. Let $(w, \alpha)_\infty$ be the corresponding point of $(w, \alpha)$ in the compactification. (See Figures 1, 2, and 3.)

A $\mathbb{Z}^n$-lattice is the set of points

$$Z_n^\mathbb{Z} = \{[x_1,x_2,...,x_n]^T \in \mathbb{R}^n | \text{at least } n - 1 \text{ of } \{x_1,x_2,...,x_n\} \text{ is an integer} \}.$$ 

The points $\alpha \in \mathbb{Z}^n \subset Z_n^\mathbb{Z}$ are called the lattice points of $Z_n^\mathbb{Z}$. Suppose that $\alpha$ and $\alpha'$ are lattice points of $Z_n^\mathbb{Z}$ such that $\alpha_j = \alpha'_j$ for $j \neq i$ and $\alpha_i = \alpha_i' - 1$. Then the open arc in $Z_n^\mathbb{Z}$ from $\alpha$ to $\alpha'$ will be represented by $(\alpha_i, \alpha_i + 1)_{i=1}^{\alpha_{i}+1} = (\alpha_i' - 1, \alpha_i')_{i=1}^{\alpha_{i}'}$. The compactification of $Z_n^\mathbb{Z}$ by identifying $\infty$ with a point will be denoted by $Z_n^\mathbb{Z}/\infty$.

**Theorem 5.** Let $G : Z(k,n) \to Z(k,n)$ be a homeomorphism. Then there exists a monotone map $\phi : Z(k,n) \to Z_n^\mathbb{Z}$ and a homeomorphism $g : Z_n^\mathbb{Z} \to Z_n^\mathbb{Z}$ such that $g \circ G = g \circ \phi$ where

$$\phi(W_\alpha(k,n)) = \alpha \cup \bigcup_{i=1}^n (\alpha_i, \alpha_i + 1/2)_{i=1}^{\alpha_i+1} \cup \bigcup_{i=1}^n (\alpha_i - 1/2, \alpha_i)_{i=1}^{\alpha_i}.$$
Proof. Define $\phi$ in the following way:
\[ \phi(x) = \begin{cases} 
\alpha & \text{if } x \in \hat{X}_\alpha \\
\alpha_i + 1/2 - \tan^{-1}(x)/\pi \in (\alpha_i, \alpha_i + 1/2]_{i+n} & \text{if } x \in R_{i+n}^\alpha \\
\alpha_i - 1/2 - \tan^{-1}(x)/\pi \in [\alpha_i - 1/2, \alpha_i) & \text{if } x \in R_i^\alpha
\end{cases} \]
Then define $g = \phi G \circ \phi^{-1}$. Then it is easily checked that $\phi$ and $g$ have the prescribed properties. □

Corollary 6. Let $\tilde{G} : Z(k, n)/\infty \to Z(k, n)/\infty$ be a homeomorphism. Then there exists a monotone map $\tilde{\phi} : Z(k, n)/\infty \to Z^\alpha_{L}/\infty$ and a homeomorphism $\tilde{g} : Z^\alpha_{L}/\infty \to Z^\alpha_{L}/\infty$ such that $\tilde{\phi} \circ \tilde{G} = \tilde{g} \circ \tilde{\phi}$ where

\[ \tilde{\phi}(W_\alpha(k, n)) = \alpha \cup \left( \bigcup_{i=1}^{n} (\alpha_i, \alpha_i + 1/2]_{i+n} \right) \cup \left( \bigcup_{i=1}^{n} [\alpha_i - 1/2, \alpha_i) \right) \]

and $\tilde{\phi}(\infty) = \infty$.

Suppose that $f_1, f_2, ..., f_n$ are a set of homeomorphisms of $Z^\alpha_L$, $Z^\alpha_L/\infty$, $Z(k, n)$, or $Z(k, n)/\infty$. Then define

\[ f^\gamma = f_1^{\gamma_1} \circ f_2^{\gamma_2} \circ ... \circ f_n^{\gamma_n} \]
where \( \gamma = [\gamma_1 \gamma_2 \ldots \gamma_n]^T \in \mathbb{Z}^n \). Likewise, if \( q = [q_1 q_2 \ldots q_m]^T \) then define
\[
[q]^\gamma = f_1^{q_1 \gamma_1} \circ f_2^{q_2 \gamma_2} \circ \ldots \circ f_n^{q_n \gamma_n}.
\]

**Theorem 7.** \( Z(k,n) / \infty \) admits an \( \mathbb{Z}^{n+1} \) expansive action.

**Proof.** Let \( h \) be the expansive homeomorphism on \( W(k,n) \) and define \( H : Z(k,n) \rightarrow Z(k,n) \) by \( H(z,\alpha) = (h(z),\alpha) \). Then for \( i \in \{1,\ldots,n\} \) define \( F_i : Z(k,n) \rightarrow Z(k,n) \) by
\[
F_i(z,\alpha) = F_i(z,\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n) = (z,\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n).
\]
Let \( \tilde{H}, \tilde{F}_1, \ldots, \tilde{F}_n \) be the corresponding extensions of \( H, F_1, \ldots, F_n \) on \( Z(k,n) / \infty \). Clearly, \( H, F_1, \ldots, F_n \) and hence \( \tilde{H}, \tilde{F}_1, \ldots, \tilde{F}_n \) commute.

Pick \( \beta \in \mathbb{Z}^n \) and consider \( W_\beta(k,n) \). \( \tilde{H} |_{W_\beta(k,n)} \) is an expansive homeomorphism of \( W_\beta(k,n) \). Let the expansive constant for \( \tilde{H} |_{W_\beta(k,n)} \) be \( c_1 \) and
\[
Q = \bigcup_{i=1}^n \{ (\beta_i + 1/2, \beta_i + 3/4)_{i+1}^2 \} \cup \bigcup_{i=1}^n \{ (\beta_i - 3/4, \beta_i - 1/2)_{i+1}^2 \}.
\]
Then define
\[
c_2 = d(Z(k,n) / \infty - W_\beta(k,n) \cup \tilde{x}^{-1}(Q), W_\beta(k,n)).
\]
Let \( c = \min\{c_1,c_2\} \). It will be shown that \( c \) is the expansive constant for \( \tilde{H}, \tilde{F}_1, \ldots, \tilde{F}_n \).

Let \( x, y \) be distinct points of \( Z(k,n) / \infty \). There are 3 cases:

**Case 1:** \( x = (x,\alpha) \) and \( y = (y,\alpha) \).

Then \( x \neq y \). Let \( \gamma = \beta - \alpha \). Then \( \tilde{F}\gamma(x,\alpha) \) and \( \tilde{F}\gamma(y,\alpha) \) are distinct points of \( W_\beta(k,n) \). Hence there exist an integer \( n \) such that \( d(\tilde{H}^n \circ \tilde{F}\gamma(x,\alpha), \tilde{H}^n \circ \tilde{F}\gamma(y,\alpha)) \geq c_1 \geq c \).

**Case 2:** \( x = (x,\alpha) \) and \( y = (y,\alpha') \) where \( \alpha \neq \alpha' \).

Again let \( \gamma = \beta - \alpha \). Then \( \tilde{F}\gamma(x,\alpha) \in W_\beta(k,n) \) and \( \tilde{F}\gamma(y,\alpha') \notin W_\beta(k,n) \). Hence there exists \( n \in \mathbb{Z} \) such that \( \tilde{H}^n \circ \tilde{F}\gamma(x,\alpha) \in W_\beta(k,n) \) and \( \tilde{H}^n \circ \tilde{F}\gamma(y,\alpha') \notin \tilde{x}^{-1}(Q) \cup W_\beta(k,n) \). Thus it follows that \( d(\tilde{H}^n \circ \tilde{F}\gamma(x,\alpha), \tilde{H}^n \circ \tilde{F}\gamma(y,\alpha')) \geq c_2 \geq c \).

**Case 3:** \( x = (x,\alpha) \) and \( y = \infty \).

Proof is similar to Case 2.

A translation \( f : \mathbb{Z}^n_1 \rightarrow \mathbb{Z}^n_2 \) is a homeomorphism of the form \( f(x) = x + d \) where \( d \in \mathbb{Z}^n \). Also if \( x \in \mathbb{Z}^n \) and \( V \) is a proper subspace of \( \mathbb{Z}^n \), then define \( \text{dist}(x,V) = \inf\{ \| x - v \| \mid v \in V \} \).

**Theorem 8.** Let \( f_1,f_2,\ldots,f_n \) be translations of \( \mathbb{Z}^n \) and \( \{\alpha_1,\alpha_2,\ldots,\alpha_m\} \) be a finite set of lattice points. Then there exists a lattice point \( y \) such that \( f_1^\gamma(y) \in \{\alpha_1,\ldots,\alpha_m\} \) for at most a finite number of distinct \( \gamma \in \mathbb{Z}^n \).

**Proof.** Let \( f_i(x) = x + d_i \). Then for \( \gamma = [\gamma_1 \ldots \gamma_n]^T \) we have
\[
f_\gamma(x) = x + \gamma_1 d_1 + \gamma_2 d_2 + \ldots + \gamma_n d_n = x + [d_1 d_2 \ldots d_n] \gamma = x + A \gamma.
\]
Notice that \( A \) is a linear operator on \( \mathbb{Z}^n \). Thus, if \( A \) is invertible, then \( A \) is one-to-one. So for every \( y \in \mathbb{Z}^n \) there exists at most \( m \) vectors \( \gamma \in \mathbb{Z}^n \) such that \( f_\gamma(y) \in \{\alpha_1,\ldots,\alpha_m\} \).
On the other hand, if $A$ is not invertible, then $\text{rank}(A) < n$. Hence the range of $A$, denoted $\text{rng}(A)$, is a proper subspace of $\mathbb{Z}^n$. Thus, there exists $y \in \mathbb{Z}^n$ such that
\[
\text{dist}(y, \text{rng}(A)) > \max_{0 \leq i \leq n} \{\text{dist}(\alpha_i, \text{rng}(A))\}.
\]

It now follows that $f^\gamma(y) \notin \{\alpha_1, ..., \alpha_m\}$ for every $\gamma \in \mathbb{Z}^n$. □

**Theorem 9.** Suppose that $f$ is a homeomorphism of $\mathbb{Z}^n$. Then there exists an integer $q$ and $v \in \mathbb{Z}^n$ such that $f^q(x) = x + v$.

**Proof.** If $f$ is a homeomorphism of $\mathbb{Z}^n$, then $f$ must be of the form $f(x) = Ax + d$ where $d \in \mathbb{Z}^n$ and $A = [\pm e_{\rho(1)} \pm e_{\rho(2)} ... \pm e_{\rho(n)}]$. Here, $e_1, ..., e_n$ are the standard basis vectors for $\mathbb{Z}^n$ and $\rho$ is a permutation of $\{1, ..., n\}$. Then there exists an integer $q$ such that $A^q = I$. Let
\[
v = A^{q-1}d + A^{q-2}d + ... + Ad + d.
\]

Then
\[
f^q(x) = A^q x + A^{q-1}d + A^{q-2}d + ... + Ad + d = x + v.
\]

**Lemma 10.** Let $\tilde{F}_1, ..., \tilde{F}_n$ be a commuting collection of homeomorphisms on $Z(k,n)_{\infty}$ and $\{\alpha_1, ..., \alpha_m\}$ be a finite subset of $\mathbb{Z}^n$. Then there exists $q, \beta \in \mathbb{Z}^n$ such that
\[
[\tilde{F}^q]^{\gamma}(W_\beta(k,n)) \notin \{W_{\alpha_1}(k,n), ..., W_{\alpha_n}(k,n)\}
\]

for all but a finite number of $\gamma \in \mathbb{Z}^n$.

**Proof.** Follows from Theorem 5, Corollary 6, Theorem 8 and Theorem 9. □

**Proposition 11.** $f_1, ..., f_n$ is an expansive $\mathbb{Z}^n$ action on continuum $X$ if and only if $f_1^{q(1)}, ..., f_n^{q(n)}$ is an expansive $\mathbb{Z}^n$ action on continuum $X$ where $q(1), q(2), ..., q(n) \in \mathbb{Z}$.

**Proof.** Suppose that $f_1, f_2, ..., f_n$ is an expansive $\mathbb{Z}^n$ action on continuum $X$ with expansive constant $c$. By uniform continuity there exists a $\delta > 0$ such that if $d(x, y) \geq c$ then
\[
d(f_1^{i_1} \circ f_2^{i_2} \circ ... \circ f_n^{i_n}(x), f_1^{i_1} \circ f_2^{i_2} \circ ... \circ f_n^{i_n}(y)) \geq \delta
\]

for every choice of
\[
i_1 \in \{0, ..., q(1)\}, i_2 \in \{0, ..., q(2)\}, ..., i_n \in \{0, ..., q(n)\}.
\]

Let $x, y$ be distinct points of $X$. Then there exist integers $\gamma_1, ..., \gamma_n$ such that
\[
d(f_1^{i_1} \circ f_2^{i_2} \circ ... \circ f_n^{i_n}(x), f_1^{\gamma_1} \circ f_2^{\gamma_2} \circ ... \circ f_n^{\gamma_n}(y)) \geq c
\]

However, there then exist $i_1, ..., i_n$ as defined previously and integers $p_1, ..., p_n$ such that $\gamma_j + i_j = p_j q(j)$. So
\[
d((f_1^{q(1)})^{p_1} \circ ... \circ f_n^{q(n)})^{p_n}(x), (f_1^{q(1)})^{p_1} \circ ... \circ (f_n^{q(n)})^{p_n}(y))
\]
\[
= d(f_1^{p_1 q(1)} \circ ... \circ f_n^{p_n q(n)}(x), f_1^{p_1 q(1)} \circ ... \circ f_n^{p_n q(n)}(y))
\]
\[
= d(f_1^{\gamma_1 + i_1} \circ ... \circ f_n^{\gamma_n + i_n}(x), f_1^{\gamma_1 + i_1} \circ ... \circ f_n^{\gamma_n + i_n}(y))
\]
\[
= d(f_1^{i_1} \circ ... \circ f_n^{i_n}(f_1^{\gamma_1} \circ ... \circ f_n^{\gamma_n}(x)), f_1^{i_1} \circ ... \circ f_n^{i_n}(f_1^{\gamma_1} \circ ... \circ f_n^{\gamma_n}(y)))
\]
\[
> \delta.
\]
The proof of the converse is obvious. □

**Theorem 12.** $Z(k,n)/\infty$ does not admit an $\mathbb{Z}^n$ expansive action.

*Proof.* For purposes of a contradiction suppose that $\tilde{F}_1, \ldots, \tilde{F}_n$ is an expansive $\mathbb{Z}^n$ action on $Z(k,n)/\infty$ with expansive constant $c$. Let $B(c/2, \infty)$ be an open ball with radius $c/2$ centered at $\infty$. Then there exists a finite subset $\{\alpha_1, \ldots, \alpha_m\}$ of $\mathbb{Z}^n$ such that $W_{\alpha}(k,n) \subseteq B(c/2, \infty)$ for every $\alpha \in \mathbb{Z}^n = \{\alpha_1, \ldots, \alpha_m\}$. By Lemma 10, there exist $\beta, q \in \mathbb{Z}^n$ such that $[F^q]^{\gamma} (W_{\beta}(k,n)) \subseteq B(c/2, \infty)$ for all but a finite subset, say $\{\gamma_1, \ldots, \gamma_p\}$, of $\mathbb{Z}^n$. By uniform continuity, there exist distinct $(x, \beta), (y, \beta) \in W_{\beta}(k,n)$ such that

$$d([F^q]^{\gamma_i}(x, \beta), [F^q]^{\gamma_i}(y, \beta)) < c/2$$

for each $\gamma_i \in \{\gamma_1, \ldots, \gamma_p\}$. Therefore,

$$d([\tilde{F}^q]^{\gamma}(x, \beta), [\tilde{F}^q]^{\gamma}(y, \beta)) < c/2$$

for every $\gamma \in \mathbb{Z}^n$ which contradicts Proposition 11. □

3. More questions and examples

Let’s return to the question: “What continua admit expansive $\mathbb{Z}^n$ actions and what continua do not?” A starting point in examining this question is looking at known results (or unknown results for that matter) on the topology of continua that admit expansive homeomorphisms. For example, it is known that chainable continua [1] and tree-like continua [5] (and hence 1-dimensional non-separating plane continua) do not admit expansive homeomorphisms. However this is not so with expansive $\mathbb{Z}^n$ actions. The following example is a chainable continuum that admits an expansive $\mathbb{Z}^2$ action:

Let $f, g$ and $h$ be continuous functions on $[0, 1]$ defined the following way:

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 2 - 2x & \text{if } x \in (1/2, 1] \end{cases},$$

$$g(x) = \begin{cases} 3x & \text{if } x \in [0, 1/3] \\ 2 - 3x & \text{if } x \in (1/3, 2/3] \text{, and} \\ 3x - 2 & \text{if } x \in (2/3, 1] \end{cases},$$

$$h = f \circ g.$$ Then $f, g$ and $h$ are known as the 2-fold, the 3-fold and the 6-fold, respectively. Notice that $f, g$ and $h$ all commute.

**Lemma 13.** If $x, y$ are distinct elements of $I$, then there exists non-negative integers $p, q$ such that $d(f^p \circ g^q(x), f^p \circ g^q(y)) > 1/9$.

*Proof.* If $d(x, y) < 1/9$ then $x, y$ are both in one of the following intervals: $[0, 1/2], [1/3, 2/3], [1/2, 1]$. We now have 2 cases:

**Case 1:** $x, y$ are both in one of $[0, 1/2], [1/2, 1]$.

Then $d(f(x), f(y)) = 2d(x, y)$.

**Case 2:** $x, y$ are both in $[1/3, 2/3]$.

Then $d(g(x), g(y)) = 3d(x, y)$.

Now we can continually apply $f$ and $g$ to $x$ and $y$ until their image is at least 1/9 apart. □
Let $X_6$ be the inverse limit of $h$. Then $\langle x_i \rangle_{i=1}^\infty \in X_6$ provided that $x_i \in I$ and $x_i = h(x_{i+1})$ for each $i \in \mathbb{Z}^+$. Since $f$, $g$, and $h$ all commute we can define homeomorphisms $\hat{f} : X_6 \to X_6$ and $\hat{g} : X_6 \to X_6$ in the following ways:

$$\hat{f}(\langle x_i \rangle_{i=1}^\infty) = \langle f(x_i) \rangle_{i=1}^\infty$$

Also notice that $\hat{f}$ and $\hat{g}$ commute.

**Theorem 14.** $\hat{f}$, $\hat{g}$ form an expansive $\mathbb{Z}^2$ action on $X_6$.

**Proof.** Proof will show that $1/9$ is an expansive constant. Suppose that $\langle x_i \rangle_{i=1}^\infty$ and $\langle y_i \rangle_{i=1}^\infty$ are distinct elements of $X_6$. Then there exists a positive integer $k$ such that $x_k \neq y_k$. So

$$\hat{g}^{-k+1} \circ \hat{f}^{-k+1}(\langle x_i \rangle_{i=1}^\infty) = h^{-k+1}(\langle x_i \rangle_{i=1}^\infty) = \langle x_i \rangle_{i=k}^\infty$$

and

$$\hat{g}^{-k+1} \circ \hat{f}^{-k+1}(\langle y_i \rangle_{i=1}^\infty) = h^{-k+1}(\langle y_i \rangle_{i=1}^\infty) = \langle y_i \rangle_{i=k}^\infty$$

Also, by Lemma 13 there exist integers $p$ and $q$ such that $d(f^p \circ g^q(x_k), f^p \circ g^q(y_k)) > 1/9$. Thus

$$d(\hat{f}^{p-k+1} \circ \hat{g}^{-k+1}(\langle x_i \rangle_{i=1}^\infty), \hat{f}^{p-k+1} \circ \hat{g}^{-k+1}((y_i)_{i=1}^\infty)) = d(f^p \circ g^q(x_k), f^p \circ g^q(y_k))$$

$$> 1/9$$

\[\square\]

**Question 1** Suppose that $X$ is the inverse limit of the $k$-fold of the interval. Is the following true? $X$ admits an expansive $\mathbb{Z}^n$ action if and only if there exist distinct prime numbers $p$ and $q$ such that $k$ divides both $p$ and $q$.

Notice that $X_6$ in the previous example is indecomposable. If a continuum $X$ is $k$-cyclic and admits an expansive homeomorphism, then it is known that $X$ must contain an indecomposable subcontinuum [4].

**Question 2** If $X$ is a chainable (tree-like, $G$-like, $k$-cyclic, any) continuum that admits an expansive $\mathbb{Z}^n$ action, must $X$ contain an indecomposable subcontinuum?

**Question 3** Does the pseudo-arc admit an expansive $\mathbb{Z}^n$ action?

It is now known that 1-dimensional hereditarily indecomposable continua do not admit expansive homeomorphisms [6]. Finally, if $h$ is an expansive homeomorphism on a continuum then it has been shown that the entropy of $h$ is positive [2] and finite [10]. So the following question is natural to ask:

**Question 4** If $f_1, \ldots, f_n$ admits an expansive $\mathbb{Z}^n$ action on a continuum, must one of $f_1, \ldots, f_n$ have positive but finite entropy?

In the all of the examples in section 2, it can be shown that $\tilde{H}$ has positive and finite entropy while each of $\tilde{F}_1, \ldots, \tilde{F}_n$ has zero entropy.
4. References


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