The topology of continua that are approximated by disjoint subcontinua.

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Abstract

Suppose that \( \{Y_i\}_{i=1}^{\infty} \) is a collection of disjoint subcontinua of continuum \( X \) such that \( \lim_{i \to \infty} d_H(Y_i, X) = 0 \) where \( d_H \) is the Hausdorff metric. Then the following are true:

1. \( X \) is non-Suslinean.
2. If each \( Y_i \) is chainable and \( X \) is finitely cyclic, then \( X \) is indecomposable or the union of 2 indecomposable subcontinua.
3. If \( X \) is \( G \)-like, then \( X \) is indecomposable.
4. If \( \{Y_i\}_{i=1}^{\infty} \) all lie in the same ray and \( X \) is finitely cyclic, then \( X \) is indecomposable.

Key words: inverse limit, indecomposable continuum, non-Suslinean continuum

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1 Introduction

Many continua (such as the buckethandle continuum) that admit homeomorphisms with interesting dynamics (such as being continuum-wise expansive) have the property that there exist disjoint subcontinua limiting to the whole...
continuum [10]. Suppose that $X$ is a continuum (compact, connected metric space) such that there exists a one-to-one map $f : [0, \infty) \rightarrow X$ with the property that $X = \bigcap_{x=0}^{\infty} f([x, \infty))$. Such a continuum is often called “a ray limiting on itself”. In [2], Curry showed that if $X$ is a plane continuum that separates the plane into a finite number of complementary domains, then $X$ must be indecomposable. This paper generalizes his result with the following theorem:

**Theorem 1** \{\{Y_i\}_{i=1}^{\infty}\} is a collection of disjoint subcontinua of continuum $X$ such that $\lim_{i \to \infty} d_H(Y_i, X) = 0$ where $d_H$ is the Hausdorff metric. Then the following are true:

1. $X$ is non-Suslinean.
2. If each $Y_i$ is chainable and $X$ is finitely cyclic, then $X$ is indecomposable or the union of 2 indecomposable subcontinua.
3. If $X$ is $G$-like, then $X$ is indecomposable.
4. If $\{Y_i\}_{i=1}^{\infty}$ all lie in the same ray and $X$ is finitely cyclic, then $X$ is indecomposable.

Also, examples are given to show that the previous theorem is sharp.

A continuum is decomposable if it is the union of 2 of its proper subcontinua. A continuum is indecomposable if it is not decomposable. A continuum is hereditarily decomposable if every nondegenerate subcontinuum is decomposable. A continuum is

1. chainable (also known as arc-like)
2. tree-like
3. $G$-like
4. $k$-cyclic

if it is the inverse limit of

1. arcs
2. trees
3. topological graphs homeomorphic to the same graph $G$
4. topological graphs each having at most $k$ distinct simple closed curves

respectively. For more on inverse limits see [3], [5] or [9] and on these definitions see [7]. To prove the main result, we need results on the topology of inverse limits of graph continua. To obtain this, we must prove technical results on graph continua.
2 Results on Graph Continua

A continuum $X$ is a *graph continuum* if it can be expressed as the union of finitely many arcs any two of which can intersect in at most one or both of their end points. If $x \in X$, then define the *degree* of $x$, denoted $\text{deg}(x)$, to be the number $n$ such that

1. For every $\epsilon > 0$ there exists an open set $U_x$ which contains $x$ such that $\text{diam}(U_x) < \epsilon$ and $|\text{Bd}(U_x)| = n$.
2. There exists a $\delta > 0$ such that if $V_x$ is an open set which contains $x$ and $\text{diam}(V_x) < \delta$ then $|\text{Bd}(V_x)| \geq n$.

Let $\mathcal{V}$ be a finite set of $X$. $\mathcal{V}$ is a *set of vertices of $X$* if it has the following properties (called the *properties of a set of vertices*):

1. Every component of $X - \mathcal{V}$ is homeomorphic to the open interval,
2. The closure of every component of $X - \mathcal{V}$ is homeomorphic to an arc.

Notice that a set of vertices is not unique (this is different from combinatorial graphs) and for each graph continuum, there exists a set of vertices. We denote a continuum $X$ with a vertex set $\mathcal{V}$ by $(X, \mathcal{V})$. Also notice that if $\text{deg}(x) \neq 2$ then, $x \in \mathcal{V}$ for any set of vertices $\mathcal{V}$. Each component of $X - \mathcal{V}$ is called an *edge* of $(X, \mathcal{V})$.

The goal of this section is to prove the following theorems:

**Theorem 2** Let $G$ be a graph continuum with $k$ distinct simple closed curves, $p = 32k^2 + 4k + 7$ and $\{[a_i, b_i]\}_{i=1}^p$ be a disjoint collection of nondegenerate subarcs of $G$. Then there exists a collection $\{m_i\}_{i=1}^p$, where $m_i \in [a_i, b_i]$, with the following property:

If $H, K$ is a decomposition of $G$, then there exists a $j \in \{1, \ldots, p\}$ (dependent only on $H$ and $K$) such that one of the following is true:

1. $[a_j, b_j] \subset H$,
2. $[a_j, b_j] \subset K$,
3. $[a_j, m_j] \subset H$ and $[m_j, b_j] \subset K$,
4. $[a_j, m_j] \subset K$ and $[m_j, b_j] \subset H$.

**Theorem 3** Let $(G, \mathcal{V})$ be a graph continuum. If $A_1, A_2, A_3$ are arcs contained in the same edge $(a, b)$ then for every decomposition $H, K$, there exists an $i \in \{1, 2, 3\}$ such that $A_i \subset H$ or $A_i \subset K$.

These theorems will aid in applying Kuykendall's Theorem (see Theorem 22) to inverse limits to determine if a continuum is indecomposable or the union
of two indecomposable subcontinua. To prove these theorems we need several technical results:

**Proposition 4** If \((G, V)\) is a graph continuum with at least 2 simple closed curves, then each simple closed curve must have a vertex with degree of at least 3.

**PROOF.** Suppose that \(S\) is a simple closed curve of \(G\) such that every point of \(S\) has degree 2. Then \(S\) is a component of \(G\). Since, \(G\) is connected we obtain \(S = G\). So \(G\) only has one simple closed curve, which is a contradiction. 

\(V\) is a **minimal set of vertices** for \(G\) if no proper subset of \(V\) satisfies all of the properties of a set of vertices.

**Proposition 5** Let \(G\) be a graph continuum with at least 2 simple closed curves and let \(V\) be a minimal set of vertices for \(G\). Then each simple closed curve of \(G\) contains at most 1 element of \(V\) with degree 2.

**PROOF.** Suppose that \(S\) is a simple closed curve of \(G\) that has 2 vertices, say \(v_1\) and \(v_2\) of \(V\) with degree 2. By Proposition 4, \(S\) must have at least 1 vertex of degree greater than 2. Thus, there exists an \(a \in S \cap V\) where deg\((a) > 2\) such that \((a,v_i)\) is an edge of \((G, V)\) contained in \(S\) for some \(i \in \{1, 2\}\). Let \((v_i, b)\) be the other edge contained in \(S\) adjacent to \(v_i\) and notice that \(a \neq b\). Let \(V' = V - \{v_i\}\). Then it is easily checked that \(V'\) has the properties of a set of vertices. This contradicts the fact that \(V\) is a minimal set of vertices. 

If \(V\) is a set of vertices, let \(V^n\) denote the subset of \(V\) such that each element has degree \(n\).

**Corollary 6** Let \(G\) be a graph continuum and let \(V\) be a minimal set of vertices for \(G\). If \(C\) is the set of simple closed curves contained in \(G\), then \(|C| \geq |V^2|\).

**PROOF.** If \(G\) has no simple closed curves, then \(V^2\) is empty. If \(G\) has one simple closed curve, then \(V^2\) has at most 2 elements. If \(G\) has more than one simple closed curve, then the result follows from Proposition 5.

If \([a, b]\) is an arc in graph continuum \(G\), then denote \((a, b) = [a, b] - \{a, b\}\). Note that the interior of \([a, b]\) is not necessarily \((a, b)\) since there may be points in
Suppose that $Y$ is a connected set in $G$ that contains $(a, b)$ and with $a, b \not\in Y$. We say that $Y$ is a $T(a, b)$-subset of $G$ if every simple closed curve of $G$ that intersects $Y$ contains $(a, b)$.

**Proposition 7** Suppose that $Y$ is a $T(a, b)$-subset of $G$. Then $Y$ is uniquely arcwise connected.

**PROOF.** Suppose on the contrary that there exist arcs $[x, y], [x, y]' \subset Y$ such that $[x, y] \neq [x, y]'$. Then there exists a simple closed curve $S \subset [x, y] \cup [x, y]' \subset Y$. Thus $(a, b) \subset S$. However, since $S$ is closed, that implies that $a \in S \subset Y$ which is a contradiction. □

**Proposition 8** If $X$ is a graph continuum, then every edge $(a, b)$ of $(X, V)$ is a $T(a, b)$-subset of $X$.

**PROOF.** Every simple closed curve $S$ is the union of edges and vertices of $X$ of $(X, V)$. Thus if $S$ intersects an edge, it must contain that edge. □

**Proposition 9** Let $(X, V)$ be a graph continuum, $(a, b)$ be an edge of $(X, V)$ and $S$ be a simple closed curve of $X$. If $(a, b) \cap S \neq \emptyset$, then $(a, b) \subset S$.

**PROOF.** Suppose that $(a, b)$ is not a subset of $S$. Then there exists an $x \in (a, b) - S$ and $y \in (a, b) \cap S$ such that $[x, y] \subset (a, b)$ and $[x, y] \cap S = \{y\}$. But then $\deg(y) \geq 3$ and hence $y \in V$. Thus, $(a, b)$ is not an edge of $(X, V)$ which is a contradiction. □

**Proposition 10** Let $(X, V)$ be a graph continuum, $P$ be a subcontinuum $X$ that contains all of the simple closed curves of $X$ and $V(P)$ be a minimal set of vertices for $P$ such that $V(P) \subset V$. Let $Y$ be a component of $X - V(P)$. Then one of the following must be true:

1. $Y \cap P = \emptyset$
2. $Y \cap P$ is an edge of $(P, V(P))$.

**PROOF.** Since $V(P)$ is a vertex set for $P$, every component of $P - V(P)$ is an edge for $(P, V(P))$. Since $Y$ is connected, if $Y$ intersects a component of $P - V(P)$ it must contain it. Thus $Y$ must contain an edge of $(P, V(P))$. If $Y - P = \emptyset$, then $Y$ must be a edge for $P$. So suppose that $Y - P \neq \emptyset$ and $Y \cap P$ is disconnected. Then there exist $x, z \in Y \cap P$ and $y \in Y - P$ such that
\[ [y, x] \cup [z, y] \subset Y, \ [y, x] \cap P = \{x\}, \ [y, x] \cap [z, y] = \{y\} \text{ and } [z, y] \cap P = \{z\}. \]

Let \([x, z]\) be an arc in \(P\). Then \([y, x] \cup [x, z] \cup [z, y]\) is a simple closed curve that is not contained in \(P\) which is a contradiction. Hence \(Y \cap P\) is a subset of some edge of \(P\). \(\Box\)

**Lemma 11** Let \((X, V)\) be a graph continuum, \(P\) be a subcontinuum of \(X\) that contains all of the simple closed curves of \(X\) and \(V(P)\) be a minimal set of vertices for \(P\) such that \(V(P) \subset V\). Let \(Y\) be a component of \(X - V(P)\). Then \(Y\) is a \(T(a, b)\)-subset for some \(a, b \in V\).

**PROOF.** By Proposition 10, if \(Y \cap P \neq \emptyset\) then \(Y \cap P = (a, b)\) is an edge of \((P, V(P))\). If \(S\) is a simple closed curve of \(X\) such that \(S \cap Y \neq \emptyset\), then \(S \cap (a, b) \neq \emptyset\) since \(S \subset P\). Hence, by Proposition 9, \((a, b) \subset S\) and it follows that \(Y\) is a \(T(a, b)\)-subset of \(X\). On the other hand, suppose that \(Y \cap P = \emptyset\) and that \(S\) is a simple closed curve of \(X\). Then \(Y \cap S = \emptyset\) and hence, \(Y\) is a \(T(a, b)\)-subset of \(X\). \(\Box\)

The following well-known theorem will be very useful.

**Theorem 12** [8] Boundary Bumping Theorem: If \(K\) is a component of a proper open subset \(V\) of a continuum \(X\) then \(\text{Bd}(V) \cap K \neq \emptyset\).

**Lemma 13** Let \(Y\) be a \(T(a, b)\)-subset in graph continuum \(X\) and let \(H\) be a subcontinuum of \(X\). Then \(Y \cap H\) has at most 2 components.

**PROOF.** Suppose on the contrary that there exist 3 pairwise disjoint sets \(H_1, H_2, H_3\) such that each is the union of components of \(Y \cap H\) and such that \(Y \cap H = \bigcup_{i=1}^3 H_i\).

**Claim:** At least 2 of the following must be true:

1. There exists \(x_1 \in \text{bd}(H_1)\) and \(y_2 \in \text{bd}(H_2)\) such that \([x_1, y_2] \subset H \cap (X - Y)\)
2. There exists \(x_2 \in \text{bd}(H_2)\) and \(y_3 \in \text{bd}(H_3)\) such that \([x_2, y_3] \subset H \cap (X - Y)\)
3. There exists \(x_3 \in \text{bd}(H_3)\) and \(y_1 \in \text{bd}(H_1)\) such that \([x_3, y_1] \subset H \cap (X - Y)\).

Suppose on the contrary that 2) and 3) are false. Pick \(x \in H_3\) and \(y \in H_2\). Since \(H\) is arcwise connected, there exists an arc \([x, y] \subset H\). Then by the Boundary Bumping Theorem, \([x, y] \cap \text{Bd}(H_3) \neq \emptyset\) and \([x, y] \cap (\text{Bd}(H_1) \cup \text{Bd}(H_2)) \neq \emptyset\). Thus, there exists a subarc \([x', y']\) of \([x, y]\) such that \([x', y'] \cap \overline{H_3} = \{x'\} \subset \text{Bd}(H_3)\) and
Without loss of generality, suppose that 1) and 2) are true in the previous claim and let $H'_1, H'_2, H''_2$ and $H'_3$ be components of $H_1, H_2$ and $H_3$ such that $x_1 \in \text{Bd}(H'_1)$ and $y_2 \in \text{Bd}(H''_2)$, $x_2 \in \text{Bd}(H''_2)$ and $y_3 \in \text{Bd}(H'_3)$. (Note that $H'_2, H''_2$ are not necessarily distinct.) Then there exist arcs $[z_1, z_2], [z'_2, z_3] \subset Y$ such that

(1) $H'_1 \cap [z_1, z_2] = \{z_1\}$
(2) $H''_2 \cap [z_1, z_2] = \{z_2\}$
(3) $H'_2 \cap [z'_2, z_3] = \{z'_2\}$
(4) $H'_3 \cap [z'_2, z_3] = \{z_3\}$.

Furthermore, there exist half open arcs

(1) $(x_1, z_1) \subset H'_1$
(2) $(y_2, z_2) \subset H''_2$
(3) $(x_2, z'_2) \subset H'_2$
(4) $(y_3, z'_2) \subset H'_3$.

Thus $[x_1, y_2] \cup (x_1, z_1) \cup [z_1, z_2] \cup [z_2, y_2]$ and $[x_2, y_3] \cup (x_2, z'_2) \cup [z'_2, z_3] \cup [z_3, y_3]$ are simple closed curves that intersect $Y$. Since $Y$ is a $T(a, b)$-subset it follows that

$$(x_1, z_1) \cup [z_1, z_2] \cup [z_2, y_2] = (a, b) = (x_2, z'_2) \cup [z'_2, z_3] \cup [z_3, y_3].$$

Thus $(x_1, z_1) \cap (x_2, z'_2) \neq \emptyset$ or $(x_1, z_1) \cap [z_3, y_3) \neq \emptyset$. Hence $H_1 \cap H_2 \neq \emptyset$ or $H_1 \cap H_3 \neq \emptyset$, which contradicts $H_1, H_2$ and $H_3$ as disjoint. □

If $\mathcal{A}$ is a collection of subsets of $X$, then define $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} A$. If $X$ is a connected space, then a collection of connected subspaces $\{H_1, ..., H_n\}$ is an $n$-decomposition if $X = \{H_1, ..., H_n\}^*$ but $\{\{H_1, ..., H_n\} - \{H_i\}\}^*$ is a proper subset of $X$ for each $i \in \{1, ..., n\}$. A 2-decomposition is simply called a decomposition.

**Lemma 14** Let $Y$ be a uniquely arcwise connected subset of a graph continuum $X$ with disjoint arcs $\{[a_i, b_i]\}_{i=1}^4$. Then for each $i$ there exists $m_i \in [a_i, b_i]$ such that for any 4-decomposition $\{H_j\}_{j=1}^4$ of $Y$ there exists an $i \in \{1, ..., 4\}$ such that at least one of the following is true:

(1) There exists a $j \in \{1, ..., 4\}$ such that $[a_i, b_i] \in H_j$
(2) There exist $j_1, j_2 \in \{1, ..., 4\}$ such that $[a_i, m_i] \in H_{j_1}$ and $[m_i, b_i] \in H_{j_2}$. 

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PROOF. Notice that since $Y$ is uniquely arcwise connected, if $a_i, b_i \in H_j$ for some $j$ then $[a_i, b_i] \subset H_j$ and 1) is satisfied. So from hereafter suppose that $a_i, b_i$ are not both in $H_j$ for any $j$.

Let $\{[p_i, m_i]\}_{i=2}^4$ be the unique arcs in $Y$ such that $[p_2, m_2] \cap [a_1, b_1] = \{p_2\}$, $[p_3, m_3] \cap ([a_1, b_1] \cup [a_2, b_2]) = \{p_3\}$, $[p_4, m_4] \cap ([a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]) = \{p_4\}$, and $[p_i, m_i] \cap [a_i, b_i] = \{m_i\}$ for each $i \in \{2, 3, 4\}$.

The following claim follows from the fact that $Y$ is uniquely arcwise connected:

**Claim:** If $[a_\alpha, b_\alpha] \cap H_j \neq \emptyset$ and $[a_\beta, b_\beta] \cap H_j \neq \emptyset$ where $\alpha < \beta$ then $[p_\beta, m_\beta] \subset H_j$ and hence $m_\beta \in H_j$.

Since $\bigcup_{j=1}^4 H_j = Y$, there exists a partition $\{P_j\}_{j=1}^4$ of $\{a_i, b_i\}_{i=1}^4$ such that $P_j \subset H_j$ for each $j$. If for some $j$ $|P_j| > 4$, then by the pigeon-hole principal there exists an $i$ such that $a_i, b_i \in H_j$ which contradicts the assumption. Thus, we may assume that $|P_j| \leq 4$ for each $j$. There are many cases. However, all are of the form of one of the following cases:

**Case 1:** $|P_1| = 4$, $|P_2| \geq 2$. Without loss of generality we may assume $P_1 = \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$ and $P_2 = \{b_{i_1}, b_{i_2}\}$ where $i_1 < i_2$. Then it follows from the claim that $m_{i_2} \in H_1$ and $m_{i_2} \in H_2$. Thus, since $Y$ is uniquely arcwise connected we have $[a_{i_2}, m_{i_2}] \subset H_1$ and $[m_{i_2}, b_{i_2}] \subset H_2$.

**Case 2:** $|P_1| = 3$, $|P_2| = 3$.

Without loss of generality we may assume $P_1 = \{a_{i_1}, a_{i_2}, a_{i_3}\}$ and $P_2 = \{b_{k_1}, b_{k_2}, b_{k_3}\}$ where $i_1 < i_2 < i_3$ and $k_1 < k_2 < k_3$. Thus, by the pigeon-hole principal, there exist $\alpha_1, \alpha_2, \alpha_3$ such that $i_{\alpha_1} = k_{\beta_1}$ and $i_{\alpha_2} = k_{\beta_2}$. Without loss of generality assume $i_{\alpha_1} < i_{\alpha_2}$. It follows from the claim that $m_{i_{\alpha_2}} \in H_1 \cap H_2$. Hence, $[a_{i_{\alpha_2}}, m_{i_{\alpha_2}}] \subset H_1$ and $[m_{i_{\alpha_2}}, b_{i_{\alpha_2}}] \subset H_2$.

**Case 3:** $|P_1| = 3$, $|P_2| = 2$, $|P_3| = 2$.

Without loss of generality we may assume $P_1 = \{a_{i_1}, a_{i_2}, a_{i_3}\}$, $P_2 = \{c_{k_1}, c_{k_2}\}$ and $P_3 = \{c_{k_3}, c_{k_4}\}$ where $i_1 < i_2 < i_3$, $k_1 < k_2$, $k_3 < k_4$, and $c_{k_i} \in \{a_{k_i}, b_{k_i}\}$. If $k_2 = k_3$ then it follows from the claim that $m_{k_2} \in H_2 \cap H_3$ and without loss of generality we may assume that $c_{k_2} = a_{k_2}$ and $c_{k_4} = b_{k_4} = b_{k_2}$ since $c_{k_2}$ and $c_{k_4}$ are distinct. Hence $[a_{k_2}, m_{k_2}] \subset H_2$ and $[m_{k_2}, b_{k_2}] \subset H_3$.

Next assume that $k_2 \neq k_3$. Thus, by the pigeon-hole principal, there exist $\alpha \in \{2, 3\}$ and $\beta \in \{2, 4\}$ such that $i_{\alpha} = k_{\beta}$. Since $i_1 < i_{\alpha}$, it follows from the
claim that \( m_i \in H_1 \cap H_{\beta/2+1} \). Hence, \([a_{i_a}, m_i] \subset H_1 \) and \([m_i, b_{i_a}] \subset H_{\beta/2+1} \) since \( a_{i_a} \) and \( c_{i_a} \) must be distinct.

**Case 4:** \(|P_1| = 2, |P_2| = 2, |P_3| = 2, |P_4| = 2\).

Then by the pigeon-hole principle there exist distinct \( j_1, j_2 \) such that \( P_{j_1} = \{c_{i_1}, a_4\} \) and \( P_{j_2} = \{c_{i_2}, b_4\} \) where \( i_1 < 4 \) and \( i_2 < 4 \). Hence it follows from the claim that \( m_4 \in H_{j_1} \cap H_{j_2} \) and thus \([a_4, m_4] \subset H_{j_1} \) and \([m_4, b_4] \subset H_{j_2} \). □

A graph continuum with no simple closed curves is a tree. If \((G, \mathcal{V})\) is a graph continuum with vertex set \( \mathcal{V} \), then let \( \mathcal{E}(\mathcal{V}) \) and \( \mathcal{C} \) be the set of edges and simple closed curves of \((G, \mathcal{V})\) respectively. The following theorem is well known:

**Theorem 15** \([4]\) If \( G \) is a tree, then \(|\mathcal{E}(\mathcal{V})| = |\mathcal{V}| - 1\).

**Corollary 16** If \((G, \mathcal{V})\) is a graph continuum, then \(|\mathcal{C}| \geq |\mathcal{E}(\mathcal{V})| - |\mathcal{V}| + 1\).

**Proof.** Suppose that \(|\mathcal{E}(\mathcal{V})| > |\mathcal{V}| - 1\). Then by Theorem 15, there exists a simple closed curve \( C_1 \) in \( G \). Let \( E_1 \) be an edge in \( C_1 \), \( G_1 = G - E_1 \) and \( \mathcal{E}_1 = \mathcal{E}(\mathcal{V}) - \{E_1\} \). Notice that \( G_1 \) is still connected. If \(|\mathcal{E}_1| = |\mathcal{V}| - 1\), then \(|\mathcal{C}| \geq 1 = |\mathcal{E}(\mathcal{V})| - |\mathcal{V}| + 1\) and we are done.

On the other hand suppose that \( G_k, \{E_1, ..., E_k\}, \{C_1, ..., C_k\} \), and \( \mathcal{E}_k \) have been found and that \(|\mathcal{E}_k| > |\mathcal{V}| - 1\). Then again by Theorem 15, there exists a simple closed curve \( C_{k+1} \) in \( G_k \). Let \( E_{k+1} \) be an edge in \( C_{k+1} \), \( G_{k+1} = G - E_{k+1} \) and \( \mathcal{E}_{k+1} = \mathcal{E}(\mathcal{V}) - \{E_{k+1}\} \). Notice that \( G_{k+1} \) is still connected.

Eventually for some \( n \), \(|\mathcal{E}_n| = |\mathcal{V}| - 1\). Thus, \(|\mathcal{E}(\mathcal{V})| - |\mathcal{V}| + 1 = n\). Furthermore, by the construction, \( E_i \) is not an edge of \( C_j \) if \( i < j \). Thus \( C_i \neq C_j \) whenever \( i \neq j \). Hence, \( \{C_1, ..., C_n\} \) is a collection of distinct simple closed curves of \( G \). Therefore

\[ |\mathcal{C}| \geq n = |\mathcal{E}(\mathcal{V})| - |\mathcal{V}| + 1.\]

□

A graph continuum \((G, \mathcal{V})\) is complete if for every distinct pair of vertices \( a, b \) there exists an edge \((a, b)\) in \((G, \mathcal{V})\). The following theorem is well known:

**Theorem 17** \([4]\) If \((G, \mathcal{V})\) is a complete graph then

\[ |\mathcal{E}(\mathcal{V})| = \frac{|\mathcal{V}|(|\mathcal{V}| - 1)}{2}.\]

**Corollary 18** Let \((G, \mathcal{V})\) be a graph continuum. Then \(|\mathcal{V}| \geq \sqrt{2|\mathcal{E}(\mathcal{V})|} \).
PROOF. Since $G$ is a graph, it must be a subgraph of the complete graph on its $|V|$ vertices. Hence,

$$|E(V)| \leq \frac{|V|(|V| - 1)}{2}.$$ 

Thus, it follows that

$$|V| \geq \sqrt{2|E(V)|}.$$ 

□

Lemma 19 Let $(G, V)$ be a graph continuum such that the $\text{deg}(v) \geq 2$ for every $v \in V$ and $V$ is a minimal vertex set. Then

$$|C| \geq \frac{1}{4}|V|,$$

where $C$ is the set of simple closed curves of $G$.

PROOF. Since every edge has 2 endpoints which are vertices we have

$$2|E(V)| = \sum_{v \in V} \text{deg}(v) \geq 3|V - V^2| + 2|V^2|.$$ 

So it follows from Corollary 16 that

$$|C| \geq |E(V)| - |V| + 1 \geq \frac{3}{2}|V| - \frac{3}{2}|V^2| + |V^2| - |V| + 1 > \frac{1}{2}|V| - \frac{1}{2}|V^2|.$$ 

Also, it follows from the fact that $V$ is minimal and Corollary 6 that $|C| \geq \frac{1}{2}|V^2|$. Thus after adding the inequalities we get

$$2|C| \geq \frac{1}{2}|V|$$

and the lemma follows.

□

Proof of Theorem 2: From Lemma 13 and Lemma 14, it suffices to show that there exists a $T(a, b)$-subset $Y$ of $G$ which contains at least 4 elements of $\{[a_i, b_i]\}_{i=1}^p$. There are 3 cases to consider:

Case 1: $G$ is a tree.
Then $G$ contains 2 points, say $a$ and $b$, that have degree 1. Thus $Y = G - \{a, b\}$
is a $T(a, b)$-subset. Since $p \geq 7$, it follows that there are at least 4 elements of $\{[a_i, b_i]\}_{i=1}^p$ contained in $Y$.

**Case 2:** $G$ contains exactly 1 simple closed curve.
Let $S$ be the unique simple closed curve of $G$ and chose $a, b \in S - \bigcup_{i=1}^p [a_i, b_i]$. Then each component of $G - \{a, b\}$ is a $T(a, b)$-subset. So by the fact that $p \geq 7$ and the pigeon-hole principle, one of the components of $G - \{a, b\}$, say $Y$, contains 4 elements of $\{[a_i, b_i]\}_{i=1}^p$.

**Case 3:** $G$ contains $k$ simple closed curves where $k \geq 2$.
Let $P$ be a maximum subcontinuum of $G$ such that $\deg(x) \geq 2$ for all $x \in P$. Let $\mathcal{V}(P)$ be a minimum vertex set for $P$. It follows from Lemma 19 that $|\mathcal{V}(P)| \leq 4k$ and from Corollary 18 that $|E(\mathcal{V}(P))| \leq 8k^2$. Let $\mathcal{Q} = \{[a_i, b_i] | [a_i, b_i] \cap \mathcal{V}(P) \neq \emptyset\}$ and $\mathcal{T} = \{[a_i, b_i]\}_{i=1}^p - \mathcal{Q}$. Since $\{[a_i, b_i]\}_{i=1}^p$ are all disjoint, we may conclude that $|\mathcal{Q}| \leq |\mathcal{V}(P)| \leq 4k$ and hence $|\mathcal{T}| = p - |\mathcal{Q}| \geq 2\cdot 3k^2 + 4k + 7 - 4k > 4(8k^2)$.

Since $\mathcal{V}(P)$ is a minimal vertex set, notice that $G - \mathcal{V}(P)$ has the same number of components as $P - \mathcal{V}(P)$ which is equal to it number of edges $|E(\mathcal{V}(P))|$. Thus the number of components of $G - \mathcal{V}(P)$ is less than or equal to $8k^2$. Hence by the pigeonhole principle, there exist distinct $\{[a_i, b_i]\}_{j=1}^4 \in \mathcal{T}$ that are all in the same component, say $Y$, of $G - \mathcal{V}(P)$. By Lemma 11, $Y$ is a $T(a, b)$-subset and the theorem follows. \qed

**Proof of Theorem 3:** Let $A_i = [a_i, b_i]$ where $a < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < b$. Suppose that $H \cap A_i \neq \emptyset$ for each $i \in \{1, 2, 3\}$. (Otherwise, $A_i \subset K$ for some $i$ and we are done.) Since edge $(a, b)$ is a $T(a, b)$-subset by Proposition 8, it follows from Lemma 13 that $H \cap (a, b)$ has at most 2 components. Hence there are 2 cases:

**Case 1:** $H \cap (a, b)$ has 1 component. Then $b_1, a_3 \in H \cap (a, b)$. Thus $[a_2, b_2] \subset H \cap (a, b) \subset H$.

**Case 2:** $H \cap (a, b)$ has 2 components. Let $H_a$ be the component such that $a \in \overline{H_a}$ and $H_b$ be the component such that $b \in \overline{H_b}$. If $H_a \cap [a_2, b_2] \neq \emptyset$ then $[a_1, b_1] \subset H_a \subset H$. Otherwise, $H_b \cap [a_2, b_2] \neq \emptyset$ which implies that $[a_3, b_3] \subset H_b \subset H$. \qed
3 Inverse limits and indecomposability

Let \( \{X_i\}_{i=1}^{\infty} \) be collection of topological spaces and \( f_i : X_{i+1} \to X_i \) be a continuous function for each \( i \). The inverse limit of \((X_i, f_i)\) is a new topological space:

\[
\hat{X} = \lim_{\leftarrow} \{X_i, f_i\}_{i=1}^{\infty} = \{ (x_i)_{i=1}^{\infty} \mid x_i \in X_i \text{ and } f_i(x_{i+1}) = x_i \}.
\]

\( \hat{X} \) has the subspace topology induced on it by \( \prod_{i=1}^{\infty} X_i \). If \( (x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty} \in \hat{X} \) then define the metric on \( \hat{X} \) by

\[
\hat{d}( (x_i)_{i=1}^{\infty}, (y_i)_{i=1}^{\infty} ) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^{i-1}}
\]

where \( d_i \) is the metric on \( X_i \) and \( \{\text{diam}(X_i)\}_{i=1}^{\infty} \) is bounded. Also, let \( \pi_i : \hat{X} \to X_i \) be the \( i \)th coordinate map.

If each \( X_i \) is a topological graph with at most \( k \) distinct simple closed curves, then \( \hat{X} \) is said to be \( k \)-cyclic or finitely cyclic. If each \( X_i \) is homeomorphic to the same topological graph \( G \), then \( \hat{X} \) is said to be \( G \)-like.

The following is the Anderson-Choquet embedding theorem:

**Theorem 20** [1] Let the compact sets \( \{M_i\}_{i=1}^{\infty} \) be subsets of a given compact metric space \( X \), and let \( f_j^i : M_j \to M_i \) be continuous surjections satisfying \( f_k^i = f_j^i \circ f_j^k \) for each \( i < j < k \). Suppose that

1. for every \( i \) and \( \delta > 0 \) there exists a \( \delta' > 0 \) such that if \( i < j \), \( p \) and \( q \) are in \( M_j \), and \( d(f_j^i(p), f_j^i(q)) < \delta \) then \( d(p, q) < \delta' \)
2. for every \( \epsilon > 0 \) there exists an integer \( k \) such that if \( p \in M_k \) then

\[
\text{diam} \left( \bigcup_{k<j} (f_j^k)^{-1}(p) \right) < \epsilon.
\]

Then, then inverse limit \( M = \lim_{\leftarrow} \{M_i, f_i\}_{i=1}^{\infty} \) is homeomorphic to \( Q = \bigcap_{i=1}^{\infty} (\bigcup_{k \leq i} M_k) \), which is the sequential limiting set of the sequence \( \{M_i\}_{i=1}^{\infty} \).

Next is a variation of the Anderson-Choquet embedding theorem which will be useful. For completeness, its proof is given in the Appendix:

**Theorem 21** Suppose that \( X \) is an 1-dimensional \( k \)-cyclic continuum such
that there exist disjoint subcontinua \( \{X_j\}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} d_H(X, X_j) = 0 \). Then there exist

1. positive numbers \( \{\epsilon_i\}_{i=1}^{\infty} \) such that \( \lim_{i \to \infty} \epsilon_i = 0 \)
2. \( Y = \lim_{\leftarrow} \{G_i, f_i\}_{i=1}^{\infty} \) where each \( G_i \) is a graph
3. disjoint subcontinua \( \{Y_j\}_{j=1}^{\infty} \) of \( Y \)

such that

1. \( Y \) is homeomorphic to \( X \)
2. each \( G_i \) has at most \( k \) simple closed curves
3. \( \{\pi_i(Y_n)\}_{i=1}^{n} \) are all disjoint in \( G_i \)
4. \( d_H(G_i, f_k^k(\pi_k(Y_n))) < \epsilon_n \) for every \( k \) and \( n \geq i \).

Furthermore, if \( Y_n \) is chainable, then \( \pi_i(Y_n) \) is an arc for each \( i \geq n \).

The following is a well-known theorem due to Kuykendall:

**Theorem 22** [6] Let \( X = \lim_{\leftarrow} \{X_n, f_n\}_{n=1}^{\infty} \) be an inverse limit of continua. \( X \) is indecomposable if and only if for each \( n \) and \( \epsilon > 0 \) there exists \( m > n \) such that if \( A_m \) and \( B_m \) are two subcontinua of \( X_m \) with \( X_m = A_m \cup B_m \), then at least one of the following is true:

1. \( d_H(f_m^m(A_m), X_n) < \epsilon \)
2. \( d_H(f_m^m(B_m), X_n) < \epsilon \).

If \( A \) and \( B \) are subsets of continuum \( G \), then let \( I(A, B) \) denote a subcontinuum of \( G \) minimal about \( A \) and \( B \). That is, \( A \cup B \subset I(A, B) \), but no proper subcontinuum of \( I(A, B) \) contains \( A \cup B \). Let \( G \) be a decomposable continuum and \( A, B \) be subsets of \( G \). We say that \( A, B \) has the decomposition containment property in \( G \) if for any decomposition \( H, K \) of \( G \), at least one of the following is true:

1. \( A \subset H \)
2. \( B \subset H \)
3. \( A \subset K \)
4. \( B \subset K \).

**Proposition 23** Suppose that sets \( A, B \) have the decomposition containment property in graph continuum \( G \) and \( P \) is a subcontinuum of \( G \) that contains \( A \cup B \). Then \( A, B \) have the decomposition containment property in \( P \).

**PROOF.** Let \( E, F \) be a decomposition of \( P \). Let

\[
H = E \cup \bigcup \{C | C \text{ is a component of } G - P \text{ such that } C \cap E \neq \emptyset\}
\]
and

\[ K = F \cup \bigcup \{ C \mid C \text{ is a component of } G - P \text{ such that } C \cap F \neq \emptyset \}. \]

Then \( H, K \) is a decomposition of \( G \). Thus, one of the following is true: \( A \subset H, B \subset H, A \subset K \) or \( B \subset K \). Without loss of generality assume \( A \subset H \). Then since \( A \subset P \), it follows that \( A \subset H \cap P = E \). \( \square \)

**Lemma 24** Let \( X = \lim \{ G_i, f_i \}_{i=1}^{\infty} \) where each \( G_i \) is a graph. Suppose \( \{ \epsilon_i \}_{i=1}^{\infty} \) is a sequence of positive numbers that converges to \( 0 \) and there exist disjoint arcs \( A_i, B_i \) of \( G_i \) with the following properties:

1. There exist decompositions \( A_i^H, A_i^K \) of \( A_i \) and \( B_i^H, B_i^K \) of \( B_i \) such that \( I(A_i^H, B_i^H) \) and \( I(A_i^K, B_i^K) \) are both arcs.
2. If \( H_i, K_i \) is any decomposition of \( G_i \), then one of the following must be true: \( A_i^H \subset H_i, A_i^K \subset K_i \), or \( B_i^H \subset H_i \) or \( B_i^K \subset K_i \).
3. If \( H_i, K_i \) is any decomposition of \( G_i \), then one of the following must be true: \( A_i^H \subset H_i, A_i^K \subset K_i \), or \( B_i^H \subset H_i \) or \( B_i^K \subset K_i \).
4. \( d_H(f_n^i(A_i), G_n) < \epsilon_i \) and \( d_H(f_n^i(B_i), G_n) < \epsilon_i \) for each \( i > n \).
5. \( d_S(f_n^i(A_i^H), I(A_i^H, B_i^H)) < \epsilon_i \), \( d_S(f_n^i(B_i^H), I(A_i^H, B_i^H)) < \epsilon_i \), \( d_S(f_n^i(A_i^K), I(A_i^K, B_i^K)) < \epsilon_i \), and \( d_S(f_n^i(B_i^K), I(A_i^K, B_i^K)) < \epsilon_i \).

Then \( X = H \cup K \) where \( H \) and \( K \) are indecomposable continua. (Note: It is possible that \( H = X \) or \( K = X \).)

**PROOF.** Let \( H_n^i = f_i^i(I(A_i^H, B_i^H)) \) and \( H_n = \bigcup_{i=1}^{\infty} H_n^i \).

Notice that

\[ f_n(H_n) = f_n(\bigcup_{i=n+1}^{\infty} H_n^i) = \bigcup_{i=n+1}^{\infty} f_n(H_n^i) = \bigcup_{i=n}^{\infty} H_n^i = H_{n-1}. \]

**Claim 1:** For every \( \epsilon > 0 \) there exists \( I_\epsilon^n \) such that \( d_H(H_n, H_n^i) < \epsilon \) for every \( i \geq I_\epsilon^n \).

Since \( H_n \) is compact there exists a cover \( \{ B(x_j, \epsilon) \}_{j=1}^{p_n} \) of \( H_n \) of \( \epsilon \)-balls such that \( x_j \in H_n \). Since \( H_n = \bigcup_{j=1}^{p_n} H_n^j \), there exists an \( i_j \) where \( d(x_j, H_n^j) < \epsilon/2 \) for each \( j \in \{1, ..., p_n\} \). Let \( I_{\max} = \max\{i_j \mid j \in \{1, ..., p_n\} \} \). By uniform continuity, there exists a \( \delta > 0 \) such that if \( d(x, y) < \delta \) then \( d(f_n^k(x), f_n^k(y)) < \epsilon/2 \) for every \( k \in \{n, ..., I_{\max} \} \). Let \( I_{\delta} \) be such that \( \epsilon_i < \delta \) for every \( i \geq I_{\delta} \) and \( I_\epsilon^n = \max\{I_{\max}, I_{\delta}\} \). Since

\[ d_S(f_n^i(A_i^H), I(A_i^H, B_i^H)) < \epsilon_i < \delta \]
for each $i \geq I^n_\epsilon$ and $j \in \{1, \ldots, p_n\}$ it follows that

$$d_S(H_n^i, H_n^j) \leq d_S(f_n^i(A_n^i), f_n^j(I(A_i, B_i))) < \epsilon/2$$

for each $i \geq I^n_\epsilon$ and $j \in \{1, \ldots, p_n\}$. Hence $H_n^i \cap B(x_j, \epsilon) \neq \emptyset$ for each $i \geq I^n_\epsilon$ and $j \in \{1, \ldots, p_n\}$. Thus, since $H_n^i \subset H_n$, it follows that $d_H(H_n, H_n^i) < \epsilon$ for every $i \geq I^n_\epsilon$.

Let $H = \liminf \{f_i, H_i\}_{i=1}^\infty$.

**Claim 2:** $H$ is indecomposable.

Given $n$ and $\epsilon$, let $i = I^n_{\epsilon/2} + 1$ as in Claim 1. Since $f_n^i$ is uniformly continuous, there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f_n^i(x), f_n^i(y)) < \epsilon/2$. Let $I_\delta$ be such that if $p \geq I_\delta$ then $\epsilon_p < \delta$. Let $j = \max\{I^n_{\epsilon/2} + 1, I_\delta\}$ and $E_j, F_j$ be a decomposition of $H_j$. Then it follows from hypothesis 2 and Proposition 23 that one of the following must be true: $A_j^H \subset E_j$, $A_j^H \subset F_j$, $B_j^H \subset E_j$, or $B_j^H \subset F_j$. Without loss of generality assume $A_j^H \subset E_j$. Thus it follows from hypothesis 5 that $d_S(f_n^i(E_j), I(A_j^H, B_j^H)) < \delta$. Hence,

$$d_S(f_n^i(E_j), H_n^j) = d_S(f_n^i(E_j), f_n^j(I(A_j^H, B_j^H))) < \epsilon/2.$$

Since $f_n^i(E_j) \subset H_n$ and $d_H(H_n, H_n^j) < \epsilon/2$ it follows that

$$d_H(f_n^i(E_j), H_n) < \epsilon/2 + \epsilon/2.$$

Therefore it follows from Theorem 22 that $H$ is indecomposable.

Next let $K_n^i = f_n^i(I(A_i^K, B_i^K))$, $K_n = \bigcup_{i=n+1}^\infty K_n^i$ and $K = \lim f_i$. Then by a similar argument as for $H$, we may conclude that $K$ is indecomposable.

**Claim 3:** $X = H \cup K$.

Pick $x_n \in G_n$. Notice that by hypothesis 4) for every $\epsilon > 0$ there exists $I^n_\epsilon > 0$ such that if $i \geq I^n_\epsilon$ then $d_H(f_n^i(A_i), G_n) < \epsilon$. Thus, it follows that $d_H(H_n^i \cup K_n^i, x_n) < \epsilon$. Hence $x_n$ is a limit point of $H_n \cup K_n$. Since $H_n \cup K_n$ is compact it follows that $x_n \in H_n \cup K_n$ and thus $G_n = H_n \cup K_n$. Therefore, $X = H \cup K$. □

**Lemma 25** Let $X = \liminf \{G_i, f_i\}_{i=1}^\infty$ where each $(G_i, V_i)$ is a graph continuum. Suppose $\{\epsilon_i\}_{i=1}^\infty$ is a sequence of positive numbers that converges to 0 and
for each $i$ there exists an edge $E_i$ of $(G_i, V_i)$ that contains 3 disjoint arcs $A^i_1, A^i_2, A^i_3$ such that $d_H(f^i_n(A^i_j), G_n) < \epsilon_i$ for each $j \in \{1, 2, 3\}$. Then $X$ is indecomposable.

**PROOF.** From Theorem 3 it follows that if $H_i, K_i$ is a decomposition of $G_i$, then $A^i_j \subset H_i$ or $A^i_j \subset K_i$ for some $j \in \{1, 2, 3\}$. Thus $d_H(f^i_n(H_i), G_n) < \epsilon_i$ or $d_H(f^i_n(K_i), G_n) < \epsilon_i$. Hence, $X$ is indecomposable by Theorem 22. \qed

4 Main Results

In this section, we prove the main results of the paper. Let $B(y, \epsilon)$ be the open $\epsilon$-ball centered at $y$.

**Proposition 26** Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are maps. Let $U$ be a subset of $Z$, $V$ be a component of $g^{-1}(U)$ and $C$ be component of $f^{-1}(V)$. Then $C$ is a component of $(g \circ f)^{-1}(U)$.

**PROOF.** Clearly $C \subset (g \circ f)^{-1}(U)$, so suppose that $C'$ is a component of $(g \circ f)^{-1}(U)$ such that $C$ is a proper subset of $C'$. If $f(C') \subset V$ then $C$ is properly contained in a connected subset of $f^{-1}(V)$ and it follows that $C$ is not a component of $f^{-1}(V)$. Hence, $f(C') \not\subset V$. However, $V \cap f(C') \neq \emptyset$. Since $f$ is continuous, $f(C')$ and thus $f(C') \cup V$ are connected. But $g(f(C') \cup V) \subset U$. Therefore, $f(C') \cup V$ is a connected subset of $g^{-1}(U)$ that properly contains $V$. However, this contradicts the fact that $V$ is a component of $g^{-1}(U)$. Thus, $C$ is not a proper subset of $C'$. Hence $C$ must be a component of $(g \circ f)^{-1}(U)$. \qed

Let $G$ be a graph continuum and $x \in G$. $(U, V)_x$ is a double separator for $x$ if $U, V$ are connected open sets that contain $x$ such that $V \subset U$ and $\text{Bd}(U) \neq \emptyset$. A map $f : X \rightarrow Y$ is $\epsilon$-onto if $\sup_{y \in Y}\{d(y, f(X))\} < \epsilon$.

**Lemma 27** Let $G$ be a graph continuum and $(U, V)_y$ be a double separator for some $y \in G$. Then there exists a $\delta > 0$ such that if $f_\epsilon : X \rightarrow G$ is any $\epsilon$-onto map of a continuum $X$ into $G$ with $\epsilon < \delta$, then there exists a component $C$ of $f_\epsilon^{-1}(U)$ such that $f_\epsilon(C) \cap \text{Bd}(U) \neq \emptyset$ and $f_\epsilon(C) \cap V \neq \emptyset$.

**PROOF.** First, since $V$ is open, there exists a $\delta > 0$ such that $B(y, \delta) \subset V$. Suppose that $\epsilon < \delta$ and no component of $f_\epsilon^{-1}(U)$ intersects $B(y, \delta)$. Then $d(y, f_\epsilon(X)) \geq \delta > \epsilon$ which contradicts the fact that $f_\epsilon$ is $\epsilon$-onto.
Next, let $C$ be any component of $f_\varepsilon^{-1}(U)$. Then, $\text{Bd}(f_\varepsilon^{-1}(U)) \cap C \neq \emptyset$ by the Boundary Bumping Theorem. However, if $f_\varepsilon(C) \cap \text{Bd}(U) = \emptyset$ then subcontinuum $C$ is a proper subset of open set $f_\varepsilon^{-1}(U)$ which is a contradiction. □

**Proposition 28** Let $U$ be a connected open set of a continuum $X$ such that $\text{Bd}(U) \neq \emptyset$ and $V$ be a open set such that $V \subset U$. Suppose that $Y$ is a subcontinuum of $X$ such that $Y \cap \text{Bd}(U) \neq \emptyset$ and $Y \cap V \neq \emptyset$. Then $Y \cap \text{Bd}(V) \neq \emptyset$.

**PROOF.** Suppose on the contrary that $Y \cap \text{Bd}(V) = \emptyset$. Then $Y \cap V \cap \text{Bd}(V) = \emptyset$. Thus $Y \cap V \subset V$. It follows that $Y \cap V$ and $Y - V$ are disjoint nonempty closed sets whose union is $V$. Hence $Y$ is not connected. This contradicts the fact that $Y$ is a continuum. □

Define $\mathcal{W}_n = \{(S_i)_{i=1}^n | S_i \in \{A, B\}\}$. If $W_n = \langle S_1, ..., S_n \rangle \in \mathcal{W}_n$, then define $\langle W_n, A \rangle = \langle S_1, ..., S_n, A \rangle$ and $\langle W_n, B \rangle = \langle S_1, ..., S_n, B \rangle$. Notice that $(W_n, A), (W_n, B) \in \mathcal{W}_{n+1}$. Finally, let $\mathcal{W} = \{(S_i)_{i=1}^\infty | S_i \in \{A, B\}\}$ and if $W = \langle S_i \rangle_{i=1}^\infty \in \mathcal{W}$ then let $\Pi_n(W) = \langle S_i \rangle_{i=1}^n$.

**Lemma 29** $X = \lim\{G_i, f_i\}_{i=1}^\infty$ where each $G_i$ is a graph continuum and let $\varepsilon_i \to 0$. Suppose that for each $i$ there exist disjoint subcontinua $A_i, B_i$ of $G_i$ such that $f_j^1|_{A_i}$ and $f_j^1|_{B_i}$ are $\varepsilon_i$ onto $G_j$ for every $j < i$. Then $X$ is non-Suslinean.

**PROOF.** Let $x \in G_1$ and $(U, V)_x$ be a double separator for $x$. There exists $\delta_1 > 0$ such that $B(x, \delta_1) \subset V$. Choose $i_1$ such that $\varepsilon_{i_1} < \delta_1$. There exist components $U_{(A)}^{i_1}, U_{(B)}^{i_1}$ of $(f_{i_1}^1|_{A_{i_1}})^{-1}(U)$, $(f_{i_1}^1|_{B_{i_1}})^{-1}(U)$, respectively, such that $f_{i_1}^1(U_{(A)}^{i_1}) \cap V \neq \emptyset$ and $f_{i_1}^1(U_{(B)}^{i_1}) \cap V \neq \emptyset$. Furthermore, since $A_{i_1}$ and $B_{i_1}$ are disjoint continua, it follows that $U_{(A)}^{i_1} \cap U_{(B)}^{i_1} = \emptyset$. Let $V_{(A)}^{i_1}, V_{(B)}^{i_1}$ be components of $(f_{i_1}^1|_{A_{i_1}})^{-1}(V)$, $(f_{i_1}^1|_{B_{i_1}})^{-1}(V)$, respectively, and choose $x_{(A)}^{i_1} \in V_{(A)}^{i_1}$ and $x_{(B)}^{i_1} \in V_{(B)}^{i_1}$. Then, $(U_{(A)}^{i_1}, V_{(A)}^{i_1})_{x_{(A)}^{i_1}}$ and $(U_{(B)}^{i_1}, V_{(B)}^{i_1})_{x_{(B)}^{i_1}}$ are double separators.

Continuing inductively, suppose that $i_{n-1}$ and $\{U_{W_{n-1}}^{n-1}\}_{W_{n-1} \in \mathcal{W}_{n-1}}$ have been found with the following properties:

1. each $U_{W_{n-1}}^{n-1}$ is a component of $(f_{i_{n-1}}^{i_{n-1}})^{-1}(U)$
2. $f_{i_{n-1}}^{i_{n-1}}(U_{W_{n-1}}^{n-1}) \cap V \neq \emptyset$
3. $\{U_{W_{n-1}}^{n-1}\}_{W_{n-1} \in \mathcal{W}_{n-1}}$ is a pairwise disjoint collection.

Let $V_{W_{n-1}}^{n-1} = U_{W_{n-1}}^{n-1} \cap (f_{i_{n-1}}^{i_{n-1}})^{-1}(V)$ and choose $x_{W_{n-1}}^{n-1} \in V_{W_{n-1}}^{n-1} \subset U_{W_{n-1}}^{n-1}$. Since $\mathcal{W}_{n-1}$ is finite, there exists a $\delta_{n-1} > 0$ such that $B(x_{W_{n-1}}^{n-1}, \delta_{n-1}) \subset V_{W_{n-1}}^{n-1}$.
for each $W_{n-1} \in \mathcal{W}_{n-1}$. Choose $i_n > i_{n-1}$ such that $\epsilon_{i_n} < \delta_{n-1}$. There exist components $U^n_{(W_{n-1}, A)}$, $U^n_{(W_{n-1}, B)}$ of $(f^n_{i_{n-1}}|A_{n,i})^{-1}(U^n_{W_{n-1}})$, $(f^n_{i_{n-1}}|B_{n,i})^{-1}(U^n_{W_{n-1}})$, respectively, such that $f^n_{i_{n-1}}(U^n_{(W_{n-1}, A)}) \cap V^n_{W_{n-1}} \neq \emptyset$ and $f^n_{i_{n-1}}(U^n_{(W_{n-1}, B)}) \cap V^n_{W_{n-1}} \neq \emptyset$. It follows from Proposition 26 that $U^n_{(W_{n-1}, A)}$ and $U^n_{(W_{n-1}, B)}$ are components of $f^n(U)$ and from Lemma 27 that $f^n_{i_{1}}(U^n_{(W_{n-1}, A)}) \cap V \neq \emptyset$ and $f^n_{i_{1}}(U^n_{(W_{n-1}, B)}) \cap V \neq \emptyset$. Finally, since $A_{n,i}$ and $B_{n,i}$ are disjoint subcontinua and since $\{U^n_{W_{n-1}}\}_{W \in \mathcal{W}_{n-1}}$ is a collection of pairwise disjoint subcontinua, it follows that $\{U^n_{W_{n-1}}\}_{W \in \mathcal{W}_{n}}$ is a collection of pairwise disjoint subcontinua. So the induction continues.

For each $W \in \mathcal{W}$ define $H^j_W = \bigcap_{n=j}^{\infty} f^n_{i_{j}}(U^n_{\Pi_n(W)})$. Since $\bigcap_{n=j}^{\infty} f^n_{i_{j}}(U^n_{\Pi_n(W)})$ is a nested intersection of continua, $H^j_W$ is a continuum. Thus, $H_W = \lim_{j \to \infty} H^j_W = \lim_{j \to \infty} (H^j_W \cap \text{Bd}(\mathcal{U}))$ is a subcontinuum of $X$. Since $f^n_{i_{1}}(U^n_{\Pi_n(W)}) \cap \text{Bd}(\mathcal{U}) \neq \emptyset$ and $f^n_{i_{1}}(U^n_{\Pi_n(W)}) \cap V \neq \emptyset$, we may conclude that $f^n_{i_{1}}(H^j_W) \cap \text{Bd}(\mathcal{U}) \neq \emptyset$ and that $f^n_{i_{1}}(H^j_W) \cap \text{Bd}(V) \neq \emptyset$ by Proposition 28. Thus each $H^j_W$ is non-degenerate and therefore $H_W$ is non-degenerate. Let $W' \in \mathcal{W}$ such that $W' \neq W$. Then there exists an $n$ such that $\Pi_n(W) \neq \Pi_n(W')$. Thus $U^n_{\Pi_n(W)}$ and $U^n_{\Pi_n(W')}$ are disjoint. Therefore, $H_W$ and $H_{W'}$ are disjoint. Finally, since $\mathcal{W}$ is uncountable, we may conclude that $\{H_W\}_{W \in \mathcal{W}}$ is an uncountable collection of pairwise disjoint non-degenerate continua. Hence, $X$ is non-Suslinean. □

**Theorem 30** Suppose that $\{Y_n\}_{n=1}^{\infty}$ is a collection of disjoint subcontinua of continuum $X$ such that $\lim_{n \to \infty} d_H(Y_n, X) = 0$. Then $X$ is non-Suslinean.

**PROOF.** Let $X = \lim_{i \to \infty} \{X_i\}$, $f^n_{i_{j}}$ where each $X_i$ is a graph continuum. Then

$$d_H(Y_n, X) = \sum_{i=1}^{\infty} \frac{d_H(\pi_i(Y_n), X_i)}{2^{i-1}}.$$ 

Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a collection of positive numbers that converges to 0.

There exist $N_1 > 0$ such that $d_H(Y_n, X) < \epsilon_1/2$ for all $n \geq N_1$. Hence, $d_H(\pi_1(Y_n), X_i) < \epsilon_1$ for all $n \geq N_1$. Let $Y^A_1$, $Y^B_1$ be distinct elements of $\{Y_n\}_{n \geq N_1}$. Since $Y^A_1 \cap Y^B_1 = \emptyset$, there exists an integer $k_1$ such that $\pi_{k_1}(Y^A_1) \cap \pi_{k_1}(Y^B_1) = \emptyset$. Let $A_1 = \pi_{k_1}(Y^A_1)$ and $B_1 = \pi_{k_1}(Y^B_1)$.

Continuing inductively, suppose that $k_1, \ldots, k_{i-1}$ have been found. There exists an integer $N_i$ such that $d_H(Y_n, X) < \epsilon_i/2^{k_{i-1}}$ for all $n \geq N_i$. Hence, $d_H(\pi_j(Y_n), X_j) < \epsilon_i$ for all $n \geq N_i$ and $j \leq k_{i-1}$. Let $Y^A_i$, $Y^B_i$ be distinct elements of $\{Y_n\}_{n \geq N_i}$. Again, since $Y^A_i \cap Y^B_i = \emptyset$, there exists an integer $k_i$ such that $\pi_{k_i}(Y^A_i) \cap \pi_{k_i}(Y^B_i) = \emptyset$. Let $A_i = \pi_{k_i}(Y^A_i)$ and $B_i = \pi_{k_i}(Y^B_i)$.

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Now define \( g_i = f_{k_{i-1}}^{k_i} \) and \( G_i = X_{k_i} \), where \( k_0 = 1 \). Then \( X \) is homeomorphic to \( \lim_{i \to \infty} \{ G_i, g_i \} \). Additionally, for each \( i \) there exist disjoint subcontinua \( A_i, B_i \) of \( G_i \) such that \( g_i^j|_{A_i} \) and \( g_i^j|_{B_i} \) are \( \epsilon_i \) onto \( G_j \) for every \( j < i \) since \( g_j^i(A_i) = \pi_{k_i}(Y_i^A) \) and \( g_j^i(B_i) = \pi_{k_i}(Y_i^B) \). Thus by Lemma 29, \( X \) is non-Suslinean. \( \square \)

**Theorem 31.** Suppose that \( \{ Y_i \}_{i=1}^{\infty} \) is a collection of disjoint subcontinua of continuum \( X \) such that \( \lim_{i \to \infty} d_H(Y_i, X) = 0 \) where each \( Y_i \) is chainable and \( X \) is finitely cyclic. Then \( X \) is indecomposable or the union of 2 indecomposable subcontinua.

**PROOF.** Suppose that \( X = \lim_{i \to \infty} \{ G_i, f_i \} \) where each \( (G_i, Y_i) \) is a graph continuum with vertices \( V_i \) and with at most \( k \) distinct simple closed curves. Let \( p = 32k^2 + 4k + 7 \) and let \( \{ \epsilon_i \}_{i=1}^{\infty} \) be a sequence of positive numbers that converges to 0. By Theorem 21, there exist \( k_1 > 0 \) and \( p \) disjoint arcs \( \{ A_j^p \}_{j=1}^{p} \) of \( G_{k_1} \) such that \( d_H(f_{k_1}^n(A_j), G_n) < \epsilon_i \) for each \( j \in \{ 1, ..., p \} \) and \( n < k_1 \).

Continuing inductively, suppose that \( k_1, ..., k_{i-1} \) have been found. Then by Theorem 21, there exist \( k_i > k_{i-1} \) and \( p \) disjoint arcs \( \{ A_j^p \}_{j=1}^{p} \) of \( G_{k_i} \) such that \( d_H(f_{k_i}^n(A_j), G_n) < \epsilon_i \) for each \( j \in \{ 1, ..., p \} \) and \( n < k_i \). Thus it follows from Theorems 2, 3 and Lemma 24 that \( X \) is indecomposable or the union of 2 indecomposable subcontinua. \( \square \)

**Theorem 32.** Suppose that \( \{ Y_i \}_{i=1}^{\infty} \) is a collection of disjoint subcontinua of continuum \( X \) such that \( \lim_{i \to \infty} d_H(Y_i, X) = 0 \) where \( X \) is \( G \)-like. Then \( X \) is indecomposable.

**PROOF.** Suppose that \( X = \lim_{i \to \infty} \{ G_i, f_i \} \) where \( (G, V) \) is a graph continuum. Let \( p = 2|E(V)| + |V| + 1 \) and let \( \{ \epsilon_i \}_{i=1}^{\infty} \) be a sequence of positive numbers that converges to 0. By Theorem 21, there exist \( k_1 > 0 \) and a subset \( \{ Y_j^1 \}_{j=1}^{p} \) of \( \{ Y_i \}_{i=1}^{\infty} \) such that \( \{ \pi_{k_i}(Y_j^1) \}_{j=1}^{p} \) is a collection of pairwise disjoint subcontinua of \( G_{k_1} \) such that \( d_H(f_{k_1}^n(\pi_{k_i}(Y_j^1)), G_n) < \epsilon_i \) for each \( j \in \{ 1, ..., p \} \) and \( n < k_1 \). Furthermore, it follows from the pigeon-hole principle that there exist at least three elements of \( \{ \pi_{k_i}(Y_j^1) \}_{j=1}^{p} \) that are contained in the same edge \( E_{k_1} \) (and are hence arcs).

Continuing inductively, suppose that \( k_1, ..., k_{i-1} \) have been found. Then by Theorem 21, there exist \( k_i > k_{i-1} \) and a subset \( \{ Y_j^i \}_{j=1}^{p} \) of \( \{ Y_i \}_{i=1}^{\infty} \) such that \( \{ \pi_{k_i}(Y_j^i) \}_{j=1}^{p} \) is a collection of pairwise disjoint subcontinua of \( G_{k_i} \) such that \( d_H(f_{k_i}^n(\pi_{k_i}(Y_j^i)), G_n) < \epsilon_i \) for each \( j \in \{ 1, ..., p \} \) and \( n < k_i \). Furthermore, it again follows from the pigeon-hole principle that there exist at least three elements of \( \{ \pi_{k_i}(Y_j^i) \}_{j=1}^{p} \) that are contained in the same edge \( E_{k_i} \) (and are hence arcs). Thus \( X \) is indecomposable by Lemma 25. \( \square \)
Theorem 33 Suppose $X$ is $k$-cyclic. If there exists a continuous, one-to-one into function $g : [0, \infty) \to X$ such that for every $x \in [0, \infty)$, $g([x, \infty)) = X$, then $X$ is indecomposable.

PROOF. Suppose that $X = \lim\{G_i, f_i\}_{i=1}^{\infty}$ where each $(G_i, \mathcal{V}_i)$ is a graph continuum with at most $k$ distinct simple closed curves and let $p = 32k^2 + 4k + 7$. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers that converges to 0. There exists $0 < x_1^i < y_1^i < x_2^i < y_2^i$ such that $d_H(g([x_j^i, y_j^i]), X) < \epsilon_i$ for each $j \in \{1, 2\}$.

Continuing inductively, suppose that $y_2^{i-1}$ has been found, then there exist $y_2^{i-1} < x_1^i < y_1^i < x_2^i < y_2^i$ such that $d_H(g([x_j^i, y_j^i]), X) < \epsilon_i$ for each $j \in \{1, 2\}$.

Next by Theorem 21, there exist $k_1 > 0$ and a subset $\{q(1, \alpha)\}_{\alpha=1}^{p}$ of positive integers such that

$$\{\pi_{k_1}(g([x_1^{q(1,\alpha)}, y_1^{q(1,\alpha)}]), \pi_{k_1}(g([x_2^{q(1,\alpha)}, y_2^{q(1,\alpha)}]))}_{\alpha=1}^{p}$$

is a collection of pairwise disjoint arcs of $G_k$, with the following properties:

1. $\pi_{k_1}(g([x_j^{q(1,\alpha)}, y_j^{q(1,\alpha)}])) \subset \pi_{k_1}(g([x_1^{q(1,\alpha)}, y_2^{q(1,\alpha)}]))$ for each $j \in \{1, 2\}$
2. $d_H(f_{k_1}^n(\pi_{k_1}(g([x_j^{q(1,\alpha)}, y_j^{q(1,\alpha)}]), G_n) < \epsilon_i$ for each $j \in \{1, \ldots, p\}$ and $n < k_1$.

Continuing inductively, suppose that $k_1, \ldots, k_{i-1}$ have been found. Then again by Theorem 21, there exist $k_i > k_{i-1}$ and a subset $\{q(i, \alpha)\}_{\alpha=1}^{p}$ of positive integers such that

$$\{\pi_{k_i}(g([x_1^{q(i,\alpha)}, y_1^{q(i,\alpha)}]), \pi_{k_i}(g([x_2^{q(i,\alpha)}, y_2^{q(i,\alpha)}]))\}_{\alpha=1}^{p}$$

is a collection of pairwise disjoint arcs of $G_k$, with the following properties:

1. $\pi_{k_i}(g([x_j^{q(i,\alpha)}, y_j^{q(i,\alpha)}])) \subset \pi_{k_i}(g([x_1^{q(i,\alpha)}, y_2^{q(i,\alpha)}]))$ for each $j \in \{1, 2\}$
2. $d_H(f_{k_i}^n(\pi_{k_i}(g([x_j^{q(i,\alpha)}, y_j^{q(i,\alpha)}]), G_n) < \epsilon_i$ for each $j \in \{1, \ldots, p\}$ and $n < k_i$.

Let $H_k, K_k$ be a decomposition of $G_k$. Then by Theorem 2, there exist $\alpha \in \{1, \ldots, p\}$ such that one of the following must be true:

1. $\pi_{k_i}(g([x_1^{q(i,\alpha)}, y_1^{q(i,\alpha)}])) \subset H_{k_i}$
2. $\pi_{k_i}(g([x_1^{q(i,\alpha)}, y_2^{q(i,\alpha)}])) \subset K_{k_i}$
3. $\pi_{k_i}(g([x_1^{q(i,\alpha)}, m_i])) \subset H_{k_i}$ and $\pi_{k_i}(g([m_i, y_2^{q(i,\alpha)}])) \subset K_{k_i}$ for some $m_i \in [x_1^{q(i,\alpha)}, y_2^{q(i,\alpha)}]$
4. $\pi_{k_i}(g([x_1^{q(i,\alpha)}, m_i])) \subset K_{k_i}$ and $\pi_{k_i}(g([m_i, y_2^{q(i,\alpha)}])) \subset H_{k_i}$ for some $m_i \in [x_1^{q(i,\alpha)}, y_2^{q(i,\alpha)}]$. 

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Notice that 3) implies that
\[ \pi_k(g([x_1^{q(i,\alpha)}, y_1^{q(i,\alpha)}])) \subset H_{k_i} \text{ or } \pi_k(g([x_2^{q(i,\alpha)}, y_2^{q(i,\alpha)}])) \subset K_{k_i} \]
and 4) implies that
\[ \pi_k(g([x_1^{q(i,\alpha)}, y_1^{q(i,\alpha)}])) \subset K_{k_i} \text{ or } \pi_k(g([x_2^{q(i,\alpha)}, y_2^{q(i,\alpha)}])) \subset H_{k_i}. \]
Thus for any decomposition \( H_{k_i}, K_{k_i} \) of \( G_{k_i} \) we have that \( d_{H_k}(f_{k_i}(H_{k_i}), G_n) < \epsilon_i \)
for each \( n < k_i \) or \( d_{H_k}(f_{k_i}(K_{k_i}), G_n) < \epsilon_i \). Hence, it follows from Theorem 22 that \( X \) is indecomposable. \( \square \)

Now Theorems 2-33 together give the main result Theorem 1.

5 Sharpness of results

In this section, we give examples that show that Theorem 1 is sharp.

Example 1. The following example is due to Bellamy and Krasinkiewicz [2]: Let \( K \) be the buckethandle continuum and \( A \) be the segment with end points \((1/2, 0)\) and \((1/2, 1)\). \( A \cap K \) is a Cantor middle thirds set. Identify each complementary open interval of this Cantor set with its end points to get continuum \( B \). \( B \) has the following properties:

1. \( B \) is closure of the one-to-one image of a ray.
2. \( B \) has a collection of disjoint arcs \( \{I_i\}_{i=1}^{\infty} \) such that \( \lim_{i \to \infty} d_H(I_i, B) = 0 \).
3. \( B \) is hereditarily decomposable.
4. \( B \) is not \( k \)-cyclic.

Thus Theorem 1 part 4) is sharp.

Example 2. Let \( X_n \subset \prod_{i=1}^{n} [0, 1] \) be defined by the following:

1. \( X_1 = [0, 1] \)
2. \( X_n = X_{n-1} \times \{0, 1/2^n\} \cup I_n \) where \( I_n = (1/2, 1/4, ..., 1/2^n) \times [0, 1/2^n] \).

Next define \( f_n : X_{n+1} \rightarrow X_n \) in the following way:
Define $H = \lim_{n \to \infty} \{X_n, f_n\}$. Then $H$ has the following properties:

1. $H$ has a collection of disjoint subcontinua $\{Y_i\}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} d_H(Y_i, H) = 0$.
2. $H$ is hereditarily decomposable.
3. $H$ is $k$-cyclic. In fact, $H$ is tree-like.
4. $H$ is not $G$-like.

Thus Theorem 1 parts 1) and 3) are sharp.

Example 3. Let $Y_n \subset [0, 1] \times [0, 1]$ be defined by the following:

1. $Y_1 = (\{1/2\} \times [0, 1]) \cup ([0, 1/2] \times \{1/2\})$
2. $Y_n = X_{n-1} \cup (\{2^{n-1}/2^n\} \times [0, 1]) \cup ([2^{n-1}/2^n, 2^n-1/2^n] \times \{1/2\})$.

Then let

$T_n = \{(x, y) \in Y_n | y \geq 1/2\}$

and

$B_n = \{(x, y) \in Y_n | y \leq 1/2\}$.

Next define $f_n : Y_{n+1} \to Y_n$ such that

$$f_n(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in Y_n \\ \left(\frac{2^n-1}{2^n}, 1/2\right) & \text{if } (x, y) \in [\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n+1}] \times \{1/2\} \end{cases}$$
and such that \( f_n(\left\{ \frac{2^{n-1}-1}{2^n+1} \right\} \times [0, \frac{1}{2}]) = B_n \) and \( f_n(\left\{ \frac{2^{n-1}-1}{2^n+1} \right\} \times [\frac{1}{2}, 1]) = T_n \).

Then define \( Q = \lim\{Y_n, f_n\}_{n=1}^{\infty}, T = \lim\{T_n, f_n\}_{n=1}^{\infty} \) and \( B = \lim\{B_n, f_n\}_{n=1}^{\infty} \). Then \( Q \) has the following properties:

1. \( Q \) has a collection of disjoint arc-like subcontinua \( \{A_i\}_{i=1}^{\infty} \) such that \( \lim_{i \to \infty} d_H(A_i, Q) = 0 \)
2. \( Q = T \cup B \) where \( T \) and \( B \) are both indecomposable.
3. \( Q \) is \( k \)-cyclic. In fact, \( Q \) is tree-like.
4. \( Q \) is not \( G \)-like.

Thus Theorem 1 part 2) is sharp.

\[ \begin{array}{c}
\text{Fig. 2. Continuum } Q. \\
\end{array} \]

6 Appendix: A Generalization of the Anderson-Choquet Theorem

Let \( G \) be a topological graph and \( \mathcal{V} \) be a set of vertices of \( G \). Then define

\[ B_1(G, \mathcal{V}) = B \cup \{x_E | x \text{ is the midpoint of edge } E \text{ of } G - \mathcal{V}\}. \]

Continuing inductively, define

\[ B_n(G, \mathcal{V}) = B_1(G, B_{n-1}(G, \mathcal{V})). \]

\( B_n(G, \mathcal{V}) \) is called the \( n \)th barycentric subdivision of \( (G, \mathcal{V}) \). Let \( X \) be a 1-dimensional continuum, then \( \mathcal{U} \) is a proper finite open cover of \( X \) if for every
Proposition 34 If $V$ is a collection of vertices for $G$, then $\bigcup_{n=1}^{\infty} B_n(G, V)$ is dense in $G$.

PROOF. The proposition follows from the fact that the dyadic rationals are dense in $\mathbb{R}$. □

We say that $(\mathcal{U}, \mathcal{W})$ has Property $P$ if $\mathcal{U}$ and $\mathcal{W}$ are proper finite open covers of $X$ such that

1. $\mathcal{W}$ refines $\mathcal{U}$
2. If $W \in \mathcal{W}$ and $U \in \mathcal{U}$ such that $W \cap \text{core}(U) \neq \emptyset$, then there exists a chain $[W_1, ..., W_3]$ in $\mathcal{W}$ such that $W_i \subset \text{core}(U)$ for each $i \in \{2, 3, 4\}$ and $W \in \{W_1, ..., W_5\}$
3. If $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$ then there exists a chain $[W_1, ..., W_n]$ of $\mathcal{W}$ such that $W_1 \subset \text{core}(U_1)$, $W_n \subset \text{core}(U_2)$ and $W_i \subset U_1 \cup U_2$ for each $i \in \{1, \ldots, n\}$.
4. If $U_1, U_2, U_3$ are distinct elements of $\mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$ and $U_2 \cap U_3 \neq \emptyset$, then every chain of $\mathcal{W}$ that intersects both $U_1 \cap U_2$ and $U_2 \cap U_3$ must have at least 5 links.

Proposition 35 If $(\mathcal{U}_a, \mathcal{U}_3)$ and $(\mathcal{U}_3, \mathcal{U}_a)$ satisfies part 1 and 4 of Property $P$, then $(\mathcal{U}_a, \mathcal{U}_3)$ satisfies part 1 and 4 of Property $P$.

PROOF. The fact that $(\mathcal{U}_a, \mathcal{U}_3)$ satisfies part 1 is obvious. Suppose that there exists $U_1^a$, $U_2^a$, and $U_3^a$ in $\mathcal{U}_a$ and a chain $[U_1^a, \ldots, U_4^a]$ in $\mathcal{U}_a$ that intersects both $U_3^a \cap U_2^a$ and $U_2^a \cap U_3^a$. Since $\mathcal{U}_a$ refines $\mathcal{U}_3$, for each $i \in \{1, \ldots, 4\}$, there exists $U_i^3 \in \mathcal{U}_3$ such that $U_i^3 \subset U_i^a$. However, then $\{U_i^3\}_{i=1}^{4}$ must contain a chain that intersects both $U_1^a \cap U_2^a$ and $U_3^a \cap U_2^a$ which contradicts the fact that $(\mathcal{U}_a, \mathcal{U}_3)$ satisfies part 4 of Property $P$. □

Proposition 36 Suppose that $(\mathcal{U}_a, \mathcal{U}_3)$ satisfies part 4 of Property $P$. If $U_a, U'_a \in \mathcal{U}_a$ and $U_3, U'_3 \in \mathcal{U}_a$ are such that $U_3 \cap U'_3 \neq \emptyset$ and $(U_3 \cup U'_3) \cap (U_a \cap U'_a) \neq \emptyset$ then

$$U_3 \cup U'_3 \subset \text{core}(U_a) \cup (U_a \cap U'_a) \cup \text{core}(U'_a).$$

PROOF. This follows directly from part 4 of Property $P$. □
Next define the nerve of a proper finite taut cover $\mathcal{U}$, denoted by $N(\mathcal{U})$, in the following way: For each $U_i \in \mathcal{U}$ pick $u_i \in \text{core}(U_i)$. If $U_i \cap U_j \neq \emptyset$, define the edge from $u_i$ to $u_j$ induced by $U_i, U_j$ to be the straight line segment $[u_i, u_j]$. Then define

$$N(\mathcal{U}) = \bigcup_{U_i \cap U_j \neq \emptyset} [u_i, u_j].$$

Notice that the nerves of finite taut covers of 1-dimensional covers are graph continua. The set $\{u_i | U_i \in \mathcal{U}\}$ is called the vertices induced from $N(\mathcal{U})$.

Next, if $(\mathcal{U}, \mathcal{W})$ have property $P$, construct the vertex map $f$ of $N(\mathcal{W})$ into $N(\mathcal{U})$ in the following way:

1. If $W_i \cap \text{core}(U_j) \neq \emptyset$, then have $f(w_i) = u_j$.
2. If $W_i \subset U_j \cap U_{j'}$, $W_i \cap W_{j'} \neq \emptyset$ and $W_i \cap \text{core}(U_j) \neq \emptyset$ (or $W_{j'} \cap \text{core}(U_j) \neq \emptyset$), then have $f(w_i) = 3/4u_j + 1/4u_{j'}$ (or $f(w_{j'}) = 3/4u_{j'} + 1/4u_j$).
3. If $W_i \cup W_{j'} \subset U_j \cap U_{j'}$, for every $W_{j'} \in \mathcal{W}$ such that $W_i \cap W_{j'} \neq \emptyset$, then have $f(w_i) = 1/2u_{j'} + 1/2u_j$.

Notice that $f$ maps adjacent vertices of $N(\mathcal{W})$ to either the same or to adjacent endpoints of quarter subdivisions of an edge of $N(\mathcal{U})$, i.e., the same or adjacent vertices of the second barycentric subdivision of $N(\mathcal{U})$. Extend $f$ linearly onto the edges of $N(\mathcal{W})$ producing a simplicial map $N(\mathcal{W})$ onto the second barycentric subdivision of $N(\mathcal{U})$. We call $f$ the vertex map of $N(\mathcal{W})$ into $N(\mathcal{U})$.

**Lemma 37** If $(\mathcal{U}, \mathcal{W})$ has property $P$, then the corresponding vertex map is onto.

**Proof.** Let $U_1$ and $U_2$ be distinct elements of $\mathcal{U}$ that intersect and $[u_1, u_2]$ be the corresponding nerve of $\{U_1, U_2\}$. From part 3 of Property $P$, we know there exists a chain $[W_1, ..., W_n]$ of $\mathcal{W}$ such that $W_1 \cap \text{core}(U_1) \neq \emptyset$ and $W_n \cap \text{core}(U_2) \neq \emptyset$. Thus it follows from part 2 of Property $P$ that there exists a subchain $[W_i, ..., W_j]$ of $[W_1, ..., W_n]$ such that $W_i \subset \text{core}(U_1)$, $W_j \subset \text{core}(U_2)$ and $W_k \subset (U_1 \cap U_2) \cup \text{core}(U_2)$ for each $k \in \{i, ..., j\}$. Then by the construction of the nerve map $f$, the nerve of $[W_i, ..., W_j]$ is an arc $[w_i, w_j]$ that must be mapped onto $[u_1, u_2]$ where $f(w_i) = u_1$ and $f(w_j) = u_2$. Since every edge of $\mathcal{U}$ is mapped onto by $f$, it follows that the nerve of $\mathcal{U}$ is mapped onto by $f$. □

The following 5 results all have the following hypothesis (known as Hypothesis H):

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Suppose that \( \{ \mathcal{U}_i \}_{i=1}^{\infty} \) is a collection of open covers of continuum \( X \) such that \( (\mathcal{U}_i, \mathcal{U}_{i+1}) \) has Property \( P \) for each \( i \), \( \mathcal{V}_i \) is the induced set of vertices for the nerve \( N(\mathcal{U}_i) \) and \( f_i : N(\mathcal{U}_{i+1}) \rightarrow N(\mathcal{U}_i) \) is the respective vertex map.

**Proposition 38** Suppose Hypothesis \( H \). Then \( f_k^n(\mathcal{V}_n) = B_{2n-2k}(N(\mathcal{U}_k), \mathcal{V}_k) \).

**PROOF.** This follows directly from the inductive definition of \( B_n(G, \mathcal{V}) \) and the definition of vertex map. \( \square \)

For the following results, if \( U, U' \in \mathcal{U}_i \), let \([v_U, v_{U'}]_i \) be the edge induced from \( U, U' \) in the nerve \( N(\mathcal{U}_i) \).

**Proposition 39** Suppose Hypothesis \( H \). Suppose that distinct \( U_{j+k}, U'_{j+k} \in \mathcal{U}_{j+k} \) and distinct \( U_j, U'_j \in \mathcal{U}_j \) have the following properties:

1. \( U_{j+k} \cap U'_{j+k} \neq \emptyset \)
2. \( U_j \cap U'_{j} \neq \emptyset \)
3. \( U_{j+k} \cup U'_{j+k} \subset \operatorname{core}(U_j) \cup (U_j \cap U'_j) \cup \operatorname{core}(U'_j) \)
4. \( (U_{j+k} \cup U'_{j+k}) \cap (U_j \cap U'_j) \neq \emptyset \).

Then \( f_j^{j+k}([v_{U_{j+k}}, v_{U'_{j+k}}]_{j+k}) \subset [v_U, v_{U'}]_j \).

**PROOF.** Proof is by induction on \( k \).

**Base Case:** Suppose that the hypothesis is true for \( k = 1 \).

Then the conclusion follows directly from the definition of vertex map that \( f_j([v_{U_{j+1}}, v_{U'_{j+1}}]_{j+1}) \subset [v_U, v_{U'}]_j \).

**Induction Step:** Suppose that the proposition is true for \( k = n \) and the hypothesis is true for \( k = n + 1 \).

By the fact that \( (\mathcal{U}_{j+n}, \mathcal{U}_{j+n+1}) \) has Property \( P \), there exist distinct and intersecting \( U_{j+n}, U'_{j+n} \in \mathcal{U}_i \) such that \( U_{j+n+1} \cup U'_{j+n+1} \subset \operatorname{core}(U_{j+n}) \cup (U_{j+n} \cap U'_{j+n}) \cup \operatorname{core}(U'_{j+n}) \). So, by the definition of vertex map \( f_{j+n}([v_{U_{j+n+1}}, v_{U'_{j+n+1}}]_{j+n+1}) \subset [v_{U_{j+n}}, v_{U'_{j+n}}]_{j+n} \).

Also, since \( (U_{j+n+1} \cup U'_{j+n+1}) \cap (U_j \cap U'_j) \neq \emptyset \) and \( (\mathcal{U}_{j+n}, \mathcal{U}_j) \) satisfies part 4) of Property \( P \) (by Proposition 35), it follows from Proposition 36 that

\[
U_{j+n} \cup U'_{j+n} \subset \operatorname{core}(U_j) \cup (U_j \cap U'_j) \cup \operatorname{core}(U'_j).
\]
Thus $U_{j+n}, U'_{j+n}$ and $U_j, U'_j$ satisfy the hypothesis of the proposition. Hence by the induction hypothesis it follows that

$$f_{j}^{j+n+1}([v_{U_{j+n}}, v_{U'_{j+n}}]; j+n+1) \subset f_{j}^{j+n}([v_{U_{j+n}}, v_{U'_{j+n}}]; j+n) \subset [v_{U_{j}}, v_{U'_{j}}].$$

$\square$

If $U \in \mathcal{U}$, then define $\text{adj}(U, \mathcal{U}) = \{U' \in \mathcal{U}| U \cap U' \neq \emptyset\}$.

**Corollary 40** Suppose hypothesis $H$. If $U_{j+k} \in \mathcal{U}_{j+k}$ and $U_j \in \mathcal{U}_j$ are such that $U_j \cap U_{j+k} \neq \emptyset$ then $f_{j}^{j+k}(v_{U_{j+k}}) \in N(\text{adj}(U_j, \mathcal{U}_j))$.

**PROOF.** Suppose that $U_j \cap U_{j+k} \neq \emptyset$. Then by Propositions 35 and 36, there exists a $U'_{j+k} \in \mathcal{U}_{j+k}$ and $U'_j \in \mathcal{U}_j$ such that

1. $U_{j+k} \cap U'_{j+k} \neq \emptyset$
2. $U_j \cap U'_j \neq \emptyset$
3. $U_{j+k} \cup U'_{j+k} \subset \text{core}(U_j) \cup (U_j \cap U'_j) \cup \text{core}(U'_j)$
4. $(U_{j+k} \cup U'_{j+k}) \cap (U_j \cap U'_j) \neq \emptyset$.

So it follows from Proposition 39 that

$$f_{j}^{j+k}(v_{U_{j+k}}) \in [v_{U_j}, v'_{U_j}] \subset N(\text{adj}(U_j, \mathcal{U}_j)).$$

$\square$

If $\mathcal{U}$ is a cover of continuum $X$ and $Y$ is a subset of $X$, then define

$$\mathcal{U}(Y) = \{U \in \mathcal{U}| U \cap Y \neq \emptyset\}.$$

**Lemma 41** Suppose hypothesis $H$. If $X'$ is a subcontinuum of $X$ such that $\mathcal{U}_{j+1}(X') = \mathcal{U}_{j+1}$ then $V_j \subset f_{j}^{j}(N(\mathcal{U}_i(X')))$. 

**PROOF.** Let $v_U \in V_j$. Then from part 2 of Property $P$, there exists $Q(U) \in \mathcal{U}_{j+1}$ such that

1. $Q(U) \subset \text{core}(U)$
2. if $U^j_{\alpha+1} \in \mathcal{U}_{j+1}$ such that $U^j_{\alpha+1} \cap Q(U) \neq \emptyset$, then $U^j_{\alpha+1} \subset \text{core}(U)$.

Then it follows from the construction of vertex maps that $f_{j}(N(\text{adj}(Q(U), \mathcal{U}_{j+1}))) = \{v_U\}$. Since $X' \cap Q(U) \neq \emptyset$, there exists a $U_i \in \mathcal{U}_i$ such that $U_i \cap Q(U) \neq \emptyset$. Hence, it follows from Corollary 40 that

$$f_{j+1}^{j+1}(v_{U_i}) \in N(\text{adj}(Q(U), \mathcal{U}_{j+1})).$$
Thus, $f_j^i(v_{U_i}) = v_U$ and hence $V_j \subset f_j^i(N(U_i(X')))$. \qed

**Lemma 42** Under hypothesis $H$, for every positive integer $n$ and $\epsilon > 0$ there exists a positive integer $i(n, \epsilon)$ such that
\[
\max\{d(f_j^i(V_i), G_j)|1 \leq j \leq n\} < \epsilon
\]
for every $i > i(n, \epsilon)$.

**PROOF.** By Proposition 34, for each $j \in \{1, ..., n\}$ there exists $k(j)$ such that $d(B_k(G_j), V_j) < \epsilon$ for every $k \geq k(j)$. Let $i(n, \epsilon) = \max\{k(j)|1 \leq j \leq n\} + n$. Then by Proposition 38,
\[
d(f_j^i(V_i), G_j) = d(B_{2(i-j)}(G_j), V_j, G_j) < \epsilon
\]
for each $i > i(n, \epsilon)$. \qed

Suppose that $U$ is a cover of $X$. $D \subset U$ is a **connected subcollection** if for every $A, B \in D$ there exists a chain in $D$ from $A$ to $B$. $D$ is called an **almost connected subcollection** if there exists a $C \in U$ such that $D \cup \{C\}$ is a connected subcollection. Notice that all connected subcollections are almost connected subcollections.

**Proposition 43** Suppose that $U$ is a $k$-cyclic cover and $D$ is an almost connected subcollection of $U$. Then
\[
W = (U - D) \cup \left( \bigcup_{D \in D} D \right)
\]
is a $m$-cyclic cover where $m \leq k$.

**PROOF.** Proof is by induction on $|D|$.

**Base Case:** Suppose $|D| = 2$, say $D = \{A, B\}$, and $D$ is almost connected. Then there exists $C \in U$ such that $A \cap C \neq \emptyset$ and $C \cap B \neq \emptyset$. Let $D = A \cup B$ and suppose that $[U_1, ..., U_n, D]$ is a circle-chain of $W$. Then at least one of the following is true:

1. $[U_1, ..., U_n, A]$ is a circle-chain of $U$,
2. $[U_1, ..., U_n, B]$ is a circle-chain of $U$,
3. $[U_1, ..., U_n, A, B]$ is a circle-chain of $U$,
4. $[U_1, ..., U_n, B, A]$ is a circle-chain of $U$,
5. $[U_1, ..., U_n, A, C, B]$ is a circle-chain of $U$,
6. $[U_1, ..., U_n, B, C, A]$ is a circle-chain of $U$.
Let $C(U_1, ..., U_n, D)$ be the collection of the above that are circle-chains. Suppose that $[V_1, ..., V_j, D]$ is a circle-chain of $W$ distinct from $[U_1, ..., U_n, D]$. Then $C(V_1, ..., V_j, D)$ (defined in a similar way) is nonempty and disjoint from $C(U_1, ..., U_n, D)$. Thus the number of distinct circle-chains of $U$ is greater than or equal to the number of circle-chains in $W$.

**Induction Step:** Suppose that if $U$ is a $k$-cyclic cover and $D_{n-1}$ is an almost connected subcollection of $U$ such that $|D_{n-1}| = n - 1$, then

$$W_{n-1} = (U - D_{n-1}) \cup \bigcup_{D \in D_{n-1}} D$$

is a $m_{n-1}$-cyclic cover where $m_{n-1} \leq k$.

Let $D_n$ be an almost connected subcollection of $U$ with $n$ elements, let $D'_{n-1}$ be an almost connected subcollection of $D_n$ with $n - 1$ elements and let $A$ be the unique element of $D_n - D'_{n-1}$. Define $D = \bigcup_{B \in D_{n-1}} B$. Then by the induction hypothesis, $W'_{n-1} = (U - D'_{n-1}) \cup \{D\}$ is a $m'_{n-1}$-cyclic cover where $m'_{n-1} \leq k$. Furthermore, either $A \cap D \neq \emptyset$ or there exists a $C \in U$ such that $A \cap C \neq \emptyset$ and $C \cap D \neq \emptyset$ since $D_n$ is almost connected. Thus $\{A, D\}$ is almost connected and $A \cup D = \bigcup_{B \in D_n} B$. Thus it follows from the base case that

$$W_n = (W'_{n-1} - \{A, D\}) \cup \{A \cup D\}$$

is a $m_n$-cyclic cover where $m_n \leq m'_{n-1} \leq k$. □

Suppose that $U$ is an open set and $W$ is a collection of open sets. Then define

$$A(U, W) = \{W \in W | W \subset U\}$$

and

$$CA(U, W) = \{A^* | A \text{ maximally connected subcollection of } A(U, W)\}.$$
(Otherwise, $W_0(Y)$ is a chain cover of $Y$ and we are done.) Then let

$$W_1 = (W_0 - \{A_0, B_0\}) \cup \{A_0 \cup B_0\}.$$ 

Then by Proposition 43, $W_1$ is a $m_1$-cover where $m_1 \leq k$. Also notice that $|W_1| = |W_0| - 1$.

Continuing inductively, suppose that $W_{n-1}$ and $m_{n-1}$ have been found. Suppose that there exists distinct $U_{n-1}^a, U_{n-1}^b \in \mathcal{U}(Y)$ and distinct $A_{n-1}, B_{n-1}$ and $C_{n-1}$ in $W_{n-1}(Y)$ such that

1. $A_{n-1}, B_{n-1} \subset U_{n-1}^a$ and $C_{n-1} \subset U_{n-1}^b$
2. $A_{n-1} \cap B_{n-1} = \emptyset$
3. $A_{n-1} \cap C_{n-1} \neq \emptyset$ and $B_{n-1} \cap C_{n-1} \neq \emptyset$.

(Otherwise, $W_{n-1}(Y)$ is a chain cover of $Y$ and we are done.) Then let

$$W_n = (W_{n-1} - \{A_{n-1}, B_{n-1}\}) \cup \{A_{n-1} \cup B_{n-1}\}.$$ 

Then by Proposition 43, $W_n$ is a $m_n$-cover where $m_n \leq m_{n-1}$. Again, $|W_n| = |W_{n-1}| - 1$. So this process must eventually stop, say at $p$. Then $W_p$ is a $m_p$-cover where $m_p \leq m_{p-1} \leq \ldots \leq m_1 \leq k$ and $W_p(Y)$ is a chain cover for $Y$. □

**Corollary 45** Let $X$ be a $k$-cyclic continuum and $\{X_i\}_{i=1}^n$ be a collection of pairwise disjoint chainable subcontinua of $X$. Then for every $\epsilon > 0$ there exists a proper finite $m$-cyclic proper taut open cover $W$ of $X$ where $m \leq k$ such that $\text{mesh}(W) < \epsilon$ and each $W(X_i)$ is a chain.

**PROOF.** The proof follows from inductive applications of Theorem 44. □

The following is the main proof of the Appendix.

**Theorem 46** Suppose that $X$ is an 1-dimensional $k$-cyclic continuum such that there exists disjoint subcontinua $\{X_j\}_{j=1}^\infty$ where $\lim_{j \to \infty} d_H(X, X_j) = 0$. Then there exists

1. $Y = \lim\{f_i, G_i\}_{i=1}^\infty$ where each $G_i$ is a graph
2. positive numbers $\{\epsilon_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} \epsilon_i = 0$
3. disjoint subcontinua $\{Y_j\}_{j=1}^\infty$ of $Y$

such that

1. $Y$ is homeomorphic to $X$
2. each $G_i$ has at most $k$ simple closed curves
(3) \( \{\pi_i(Y_n)\}_{i=1}^n \) are all disjoint in \( G_i \)
(4) \( d_H(G_i, f_i^k(\pi_k(Y_n))) < \epsilon_n \) for every \( k \) and \( n \geq i \).

Furthermore, if \( Y_n \) is chainable, then \( \pi_i(Y_n) \) is an arc for each \( i \geq n \).

**PROOF.** By Corollary 45 there exists a 1-dimensional proper taut open cover \( \mathcal{U}_1 \) of \( X \) such that

(1) the nerve of \( \mathcal{U}_1 \) has at most \( k \) simple closed curves
(2) \( \text{mesh}(\mathcal{U}_1) < \frac{1}{5} d(X_1, X_2) \)
(3) if \( X_1 \) and \( X_2 \) are chainable, then the nerves of \( \mathcal{U}_1(X_1) \) and \( \mathcal{U}_1(X_2) \) are arcs.

Then let

\[ 0 < \epsilon_2 < \frac{1}{3} \min\{\{d(X_k, X_m)|k \neq m \text{ and } k, m \leq 3\}\} \cup \{d(X_k, (\mathcal{U}_1 - \mathcal{U}_1(X_k))^*)|k \leq 3\} \].

Continuing inductively, suppose that \( \mathcal{U}_1, \ldots, \mathcal{U}_{n-1} \) and \( \epsilon_1, \ldots, \epsilon_n \) have been found. Then by Corollary 45 there exists a 1-dimensional proper taut open cover \( \mathcal{U}_n \) of \( X \) such that

(1) the nerve of \( \mathcal{U}_n \) has at most \( k \) simple closed curves
(2) \( \text{mesh}(\mathcal{U}_n) < \epsilon_n \)
(3) \( \mathcal{U}_n \) refines \( \mathcal{U}_{n-1} \)
(4) if \( \{X_j\}_{j=1}^{n+1} \) are all chainable, then the nerves of \( \{\mathcal{U}_n(X_j)\}_{j=1}^{n+1} \) are all arcs.

Next let

\[ 0 < \epsilon_{n+1} < \frac{1}{3} \min\{\{d(X_k, X_m)|k \neq m \text{ and } k, m \leq n+2\}\} \cup \{d(X_k, (\mathcal{U}_n - \mathcal{U}_n(X_k))^*)|k \leq n+2\} \].

Notice that in the above construction, \( \mathcal{U}_j(X_k) \cap \mathcal{U}_j^*(X_n) = \emptyset \) for \( k, n \leq j \) and

(1) \( (\mathcal{U}_j - \mathcal{U}_j(X_k))^* \cap \mathcal{U}_{j+1}^*(X_k) = \emptyset \)

for \( k \leq j + 1 \).

Let \( f_j : N(\mathcal{U}_{j+1}) \longrightarrow N(\mathcal{U}_j) \) be the induced vertex map, \( G_i = N(\mathcal{U}_i) \) and \( Y = \lim \{f_i, G_i\}_{i=1}^\infty \). It follows from the Anderson-Chochet Embedding Theorem [1] that \( Y \) is homeomorphic to \( X \).

By Lemma 42, for each \( n \), there exists a positive integer \( i(n, \epsilon_n) \) such that

\[ d(f_j^*(V_i), G_j) < \epsilon_n \] for each \( j \in \{1, \ldots, n\} \) and \( i \geq i(n, \epsilon_n) \). Since \( d_H(X_n, X) \to 0 \) as \( n \to \infty \), there exists a positive integer \( \alpha_n \) such that \( X_{\alpha_n} \cap U \neq \emptyset \) for every \( U \in \mathcal{U}_{i(n, \epsilon_n)+1} \). Therefore, by Lemma 42, \( Y_{i(n, \epsilon_n)} \subset f_{i(n, \epsilon_n)}^*(N(\mathcal{U}_i(X_{\alpha_n}))) \) for each \( i > i(n, \epsilon_n) \). It follows from (1) above that \( f_i(N(\mathcal{U}_{i+1}(X_{\alpha_n}))) \subset N(\mathcal{U}_i(X_{\alpha_n})) \).
For \( i \geq i(n, \epsilon_n) \) let \( Y^n_i = \bigcap_{j=i}^{\infty} f^j_i \left( N(U_j(X_{\alpha_n})) \right) \) and for \( i < i(n, \epsilon_n) \), let \( Y^n_i = f^{i(n, \epsilon_n)}_i \left( Y^n_i(n, \epsilon_n) \right) \). Then define \( Y^n_i = \lim \{ f_i, Y^n_i \} \). Notice that \( Y^n_i(n, \epsilon_n) \subset Y^n_i(n, \epsilon_n) \) and \( \pi_i(Y^n_i(n, \epsilon_n)) = Y^n_i(n, \epsilon_n) \). Therefore \( d(\pi_i(Y^n_i), G_i) = d(Y^n_i, G_i) < \epsilon_n \) and the rest of the properties follow.

\( \square \)

References


